ARML Competition 2017

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Head Author's Note

A little over twenty years ago, my friend and mentor Don Barry suggested that I read through a draft of the coming year's ARML contest over the winter holiday; he wanted my suggestions and feedback. I agreed, and one snowy afternoon I pulled a chair up in front of the fireplace and opened the packet he had mailed me, eager to hunker down for a few hours of challenging mathematics. You can imagine my surprise when I discovered that each "problem" had three or four candidates, so that instead of doing *one* ARML contest, I was doing four—and instead of a few hours of math, I was looking at somewhere between ten and twelve!

As I began working the problems, my initial dismay gave way to delight. While I couldn't work them all that first time through, I was immediately hooked on the challenge of creating and solving questions whose solution demanded creativity, planning, and strategy, not just brute-force calculation. I couldn't then conceive of being able to write such questions. Three years later, however—exactly seventeen years ago—inspiration struck, and my first Power Question, "Power of Association," was born. I wrote several more, and a few questions to fill in gaps in the rest of the contest, and in 2008, Don asked me to succeed him as head author. I was honored, terrified, and grateful.

Ten years and ten contests later, I'm still honored and grateful, although marginally less terrified. I've learned that writing a contest of exceptional quality requires a committee of exceptional quality, and I've been fortunate to enjoy working with the best. Our annual meeting includes math late into the night, wonderful meals and fellowship, and jokes that only the nerdiest would get—and some for which you really "had to be there." A not-insubstantial added benefit of expanding that committee has been embracing several former students as colleagues and friends.

This year marks my last as head author, although I hope and plan to continue contributing problems for many years to come. We couldn't have come this far without many people's work: among the authors, I want to particularly thank Lead Editor and perpetual Super Relay author Chris Jeuell for his unwavering eye for errors (mathematical, typographical, and LaTeXnical), unceasing patience, and unlimited generosity of spirit and good humor. (Evidence of this last is evident in this year's Super Relay, which was a wonderful parting gift.) Micah Fogel, Paul Zeitz, and the other PQ graders have provided valuable insights not just on individual Power Questions but on what makes a Power Question challenging, exciting, and feasible to grade. Don Barry and the late Bryan Sullivan trusted in me and my authorship and provided needed but always-deferential suggestions and feedback; Don, you're right that it's impossible to write a relay that's too easy. And a huge thanks goes out to all the coaches, teachers, parents, and mentors who make the contest possible for thousands of young people every year. Seeing the throngs of students taking the contest every year makes the whole thing worthwhile.

On a personal note, I want to thank my wife, Allison, and my children, Ari, Jonah, and Helen, for putting up with the many hours—often on family "vacation"—that ARML has taken. I, and they, will probably forever associate our spring break trips with late nights spent on last-minute finishing touches.

Next year's contest will take place under the leadership of George Reuter, who has long authored the NYSML contest. His accession returns us to our roots, in that ARML itself was an outgrowth of NYSML in the early 1970's. George joined our committee three years ago, and with his background in high school teaching, shares my perspective on what actual high school students can do, as well as a passion for challenging, unusual mathematics. I'm excited for his leadership, and grateful, again; but not at all terrified. Have a wonderful ride!

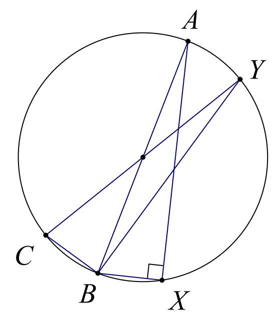
Paul J. Karafiol
April, 2017

1 Team Problems

Problem 1. Compute the number of ordered triples of positive integers (a, b, c) such that $\sqrt{a^b + c!} = 28$.

Problem 2. Chris the frog begins on a number line at 0. Chris takes jumps of lengths $1, 2, 3, \ldots, 2017$, in that order. If Chris's current location is an even integer, he jumps in the positive direction; otherwise, he jumps in the negative direction. Let $P(n)$ denote Chris's location after the n^{th} jump. Compute $\sum_{j=1}^{2017} P(j)$.

Problem 3. The diagram below shows arc $\widehat{ABC}$, which has a measure of $210°$. Points X and Y lie on the arc so that $\mathrm{m}\angle AXB = 90°$ and $\triangle ABX \cong \triangle YCB$. Given that $AX = 8$, the value of $[ABC]$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are integers and c is not divisible by any perfect square greater than 1. Compute the ordered triple (a, b, c).



Problem 4. Ari repeatedly rolls a standard, fair, six-sided die. Let $R(n)$ be the n^{th} number rolled, and let $Q(n) = R(1) \cdot R(2) \cdot \ldots \cdot R(n)$. Compute the probability that there exists an n such that $Q(n) = 100$ and for all $m < n$, $Q(m)$ is not a perfect square.

Problem 5. Given that C, A, T, F, I, S, and H are digits, not necessarily distinct, and that

$$3 \cdot \underline{C}\ \underline{A}\ \underline{T} \cdot \underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = \underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H},$$

compute the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$.

Problem 6. Let $\{a_n\}$ be a sequence with $a_0 = 1$, and for all $n > 0$, $a_n = \frac{1}{2}\sum_{i=0}^{n-1} a_i$. Compute the greatest value of n for which $a_n < 2017$.

Problem 7. On July 17, 2017, the nation of Armlandia will turn n^2 years old and the nation of Nysmlistan will turn n years old. The next four anniversaries for which Nysmlistan's age divides Armlandia's age will occur, in order, on July 17 in the years 2027, 2032, 2038, and M. Compute M.

Problem 8. Ellipse $\mathcal{E}$ has center O, major axis of length 10, and minor axis of length 4. Ellipse $\mathcal{E}'$ is obtained by rotating $\mathcal{E}$ counterclockwise about O by $60°$. Proceeding clockwise around the perimeter of $\mathcal{E}$, the intersection points of $\mathcal{E}$ and $\mathcal{E}'$ are labeled A, R, M, L. Compute $[AOL]$.

Problem 9. Let $E(n)$ denote the least integer strictly greater than n whose base-10 representation contains only even digits, and let $O(n)$ denote the least integer strictly greater than n whose base-10 representation contains only odd digits. Compute the least positive integer N for which $E(N) + O(N)$ is a multiple of 2017.

Problem 10. Compute the number of tilings of a 4×7 rectangle using only 1×1, 2×2, 3×3, and 4×4 tiles.

2 Answers to Team Problems

Answer 1. 9

Answer 2. 1

Answer 3. $(64, -32, 3)$

Answer 4. $\frac{3}{250}$ (or 0.012)

Answer 5. 6673335

Answer 6. 21

Answer 7. 2062

Answer 8. $\frac{200\sqrt{2923}}{2923}$

Answer 9. 64026

Answer 10. 1029

3 Solutions to Team Problems

Problem 1. Compute the number of ordered triples of positive integers (a, b, c) such that $\sqrt{a^b + c!} = 28$.

Solution 1. Squaring both sides of the given equation gives $a^b + c! = 784$. Accordingly, only values of c up to 6 need be considered, as $7! > 784$.

c	$784 - c!$	prime factorization of $784 - c!$
1	783	$3^3 \cdot 29$
2	782	$2 \cdot 17 \cdot 23$
3	778	$2 \cdot 389$
4	760	$2^3 \cdot 5 \cdot 19$
5	664	$2^3 \cdot 83$
6	64	2^6

For $1 \le c \le 5$, the only ordered triple satisfying the equation is $(784 - c!, 1, c)$. For $c = 6$, there are four triples: $(2, 6, 6)$, $(4, 3, 6)$, $(8, 2, 6)$, and $(64, 1, 6)$, for a total of **9** ordered triples.

Problem 2. Chris the frog begins on a number line at 0. Chris takes jumps of lengths $1, 2, 3, \ldots, 2017$, in that order. If Chris's current location is an even integer, he jumps in the positive direction; otherwise, he jumps in the negative direction. Let $P(n)$ denote Chris's location after the n^{th} jump. Compute $\sum_{j=1}^{2017} P(j)$.

Solution 2. List where Chris has landed after his first few jumps.

Jump	Before	Direction	After
1	0	+	1
2	1	−	−1
3	−1	−	−4
4	−4	+	0

In fact, this pattern continues: Chris lands back at the origin after every fourth jump. Suppose that $P(4n) = 0$. Then the next four jumps are as follows.

Jump	Before	Direction	After
$4n + 1$	0	+	$4n + 1$
$4n + 2$	$4n + 1$	−	-1
$4n + 3$	-1	−	$-4n - 4$
$4n + 4$	$-4n - 4$	+	0

This inductive argument shows that Chris lands back at the origin after every four jumps. Moreover, the sum of the four values of $P(j)$ is constant:

$$\begin{aligned}
P(4n+1) &+ P(4n+2) + P(4n+3) + P(4n+4) \\
&= (4n+1) + (-1) + (-4n-4) + 0 \\
&= -4.
\end{aligned}$$

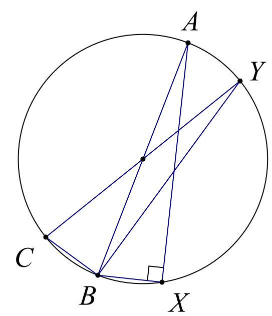
Compute $P(1) + \cdots + P(2017)$ by grouping terms in groups of four:

$$\begin{aligned}
\sum_{j=1}^{2017} P(j) &= \left(\sum_{k=0}^{503} \Big(P(4k+1) + P(4k+2) + P(4k+3) + P(4k+4)\Big)\right) + P(2017) \\
&= \left(\sum_{k=0}^{503} -4\right) + P(2017) \\
&= -2016 + P(2017).
\end{aligned}$$

Because 2016 is divisible by 4, after jump 2016, Chris will be at the origin. After jump 2017, Chris will be at 2017, so $P(2017) = 2017$, and the sum is

$$\sum_{j=1}^{2017} P(j) = -2016 + 2017 = \mathbf{1}.$$

Problem 3. The diagram below shows arc $\widehat{ABC}$, which has a measure of $210°$. Points X and Y lie on the arc so that $m\angle AXB = 90°$ and $\triangle ABX \cong \triangle YCB$. Given that $AX = 8$, the value of $[ABC]$ can be expressed in the form $a + b\sqrt{c}$, where a, b, c are integers and c is not divisible by any perfect square greater than 1. Compute the ordered triple (a, b, c).



Solution 3. Because $\angle AXB$ is a right angle, it follows that $\overline{AB}$ is a diameter of the circle containing points A, B, and C. Thus $\angle ACB$ is also right angle. Also $\angle BAC \cong \angle BYC$ because both angles subtend $\widehat{BC}$. Because $\angle BYC \cong \angle XAB$, conclude that $\triangle ABC \cong \triangle ABX$. Next, note that $m\angle XAB = m\angle BYC = \frac{1}{2} \cdot m\widehat{BC} = \frac{1}{2}(210° - 180°) = 15°$. Thus $[ABC] = [ABX] = \frac{1}{2} \cdot AX \cdot BX = \frac{1}{2} \cdot 8 \cdot (8 \tan 15°) = 32 \tan 15°$. The value of $\tan 15°$ can be computed by using the tangent half-angle identity: $\tan(\frac{x}{2}) = \frac{\sin x}{1+\cos x}$ with $x = 30°$, yielding $\tan 15° = \frac{\sin 30°}{1+\cos 30°} = \frac{1/2}{1+\sqrt{3}/2} = 2-\sqrt{3}$. Thus $[ABC] = 32(2-\sqrt{3}) = 64 - 32\sqrt{3}$, and the desired ordered triple (a, b, c) is $\mathbf{(64, -32, 3)}$.

Note: The value of $\tan 15°$ can also be computed using the identity $\tan(A - B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$, with $A = 45°$ and $B = 30°$.

Alternate Solution: Proceed as in the previous solution to conclude that $\triangle ABC \cong \triangle ABX$. Then $AB = \frac{AX}{\cos 15°}$ and $[ABC] = \frac{1}{2} \cdot AX \cdot \frac{AX}{\cos 15°} \cdot \sin 15°$. Use the subtraction identities to compute $\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° = \frac{\sqrt{6}+\sqrt{2}}{4}$ and $\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30° = \frac{\sqrt{6}-\sqrt{2}}{4}$. Dividing, $\frac{\sin 15°}{\cos 15°} = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{6-2\sqrt{12}+2}{4} = 2-\sqrt{3}$. Substituting into the expression for $[ABC]$ yields $\frac{1}{2} \cdot 8^2 \cdot \left(2 - \sqrt{3}\right) = 64 - 32\sqrt{3}$, as in the previous solution.

Problem 4. Ari repeatedly rolls a standard, fair, six-sided die. Let $R(n)$ be the n^{th} number rolled, and let $Q(n) = R(1) \cdot R(2) \cdot \ldots \cdot R(n)$. Compute the probability that there exists an n such that $Q(n) = 100$ and for all $m < n$, $Q(m)$ is not a perfect square.

Solution 4. In order for the conditions in the problem to be met, the first roll must be either 2 or 5. After the first roll, rolls of 1 can be ignored, because they are essentially equivalent to simply re-rolling; there are five other possibilities. Because of the requirement that no partial product before 100 be a perfect square, there are only five sequences of non-1 rolls that result in success: 2-5-2-5, 2-5-5-2, 5-4-5, 5-2-5-2, and 5-2-2-5. The probability of obtaining each sequence of length four is $\frac{1}{6} \cdot \left(\frac{1}{5}\right)^3 = \frac{1}{750}$, while the probability of obtaining the sequence of length three is $\frac{1}{6} \cdot \left(\frac{1}{5}\right)^2 = \frac{5}{750}$. Because there are four sequences of length four and one sequence of length three, the desired probability is $4 \cdot \frac{1}{750} + 1 \cdot \frac{5}{750} = \frac{9}{750} = \mathbf{\frac{3}{250}}$.

Problem 5. Given that C, A, T, F, I, S, and H are digits, not necessarily distinct, and that

$$3 \cdot \underline{C}\ \underline{A}\ \underline{T} \cdot \underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = \underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H},$$

compute the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$.

Solution 5. First rewrite the given equation as

$$3 \cdot \underline{C}\ \underline{A}\ \underline{T} \cdot \underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 10000 \cdot \underline{C}\ \underline{A}\ \underline{T} + \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$$

which gives

$$3 = \frac{10000}{\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}} + \frac{1}{\underline{C}\ \underline{A}\ \underline{T}}.$$

Therefore, in order to compute the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}$, and hence of $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$, $\frac{10000}{\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}}$ must be as close as possible to 3 without going over, which means $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$ must be as close to 3333 as possible. Trying $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 3334$ gives $\underline{C}\ \underline{A}\ \underline{T} = 1667$, which is impossible. Trying $\underline{F}\ \underline{I}\ \underline{S}\ \underline{H} = 3335$ gives $\underline{C}\ \underline{A}\ \underline{T} = 667$, which is the greatest possible value of $\underline{C}\ \underline{A}\ \underline{T}$.

Thus the greatest possible value for $\underline{C}\ \underline{A}\ \underline{T}\ \underline{F}\ \underline{I}\ \underline{S}\ \underline{H}$ is **6673335**.

Problem 6. Let $\{a_n\}$ be a sequence with $a_0 = 1$, and for all $n > 0$, $a_n = \frac{1}{2}\sum_{i=0}^{n-1} a_i$. Compute the greatest value of n for which $a_n < 2017$.

Solution 6. Examine the first few terms in the sequence until a pattern emerges.

$$\begin{aligned} a_1 &= \frac{1}{2} = \frac{3^0}{2^1} \\ a_2 &= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} = \frac{3^1}{2^2} \\ a_3 &= \frac{1}{2}\left(1 + \frac{1}{2} + \frac{3}{4}\right) = \frac{9}{8} = \frac{3^2}{2^3} \end{aligned}$$

The following argument shows that for $n \geq 1$, $a_n = \frac{3^{n-1}}{2^n}$. The base case is shown above. For the induction hypothesis, assume that $a_k = \frac{3^{k-1}}{2^k}$ for some $k \geq 1$; then use the geometric series sum formula with $r = \frac{3}{2}$ to compute a_{k+1}:

$$\begin{aligned} a_{k+1} &= \frac{1}{2}(a_0 + a_1 + \cdots + a_k) \\ &= \frac{1}{2}\left(1 + \frac{3^0}{2^1} + \frac{3^1}{2^2} + \frac{3^2}{2^3} + \cdots + \frac{3^{k-1}}{2^k}\right) \\ &= \frac{1}{2}\left(1 + \frac{1}{2}\left(\frac{1 - \frac{3^k}{2^k}}{1 - \frac{3}{2}}\right)\right) = \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{1 - \frac{3^k}{2^k}}{\frac{1}{2}}\right)\right) = \frac{1}{2}\left(\frac{3^k}{2^k}\right) = \frac{3^k}{2^{k+1}}. \end{aligned}$$

Thus for $n > 0$, $a_n = \frac{3^{n-1}}{2^n} = \frac{1}{2} \cdot \left(\frac{3}{2}\right)^{n-1}$.

Consequently, the number which must be computed is the greatest value of n for which $1.5^{n-1} < 2 \cdot 2017 = 4034$, i.e., $1.5^n < 6051$. Note that $1.5^4 = 2.25^2 = 5.0625 > 5$. Therefore $1.5^{20} = (1.5^4)^5 > 5^5 = 3125$, and so

$1.5^{22} > 3125 \cdot 2.25$ which is clearly larger than 6051. Thus the desired value of n is no more than 21. Moreover, because 1.5^4 is only slightly larger than 5 and $1.5^{20} = \left(1.5^4\right)^5$, one should expect that 1.5^{20} will be only slightly larger than $5^5 = 3125$. This argument suggests that $3125 \cdot 1.5 = 4627.5$ is a reasonable approximation for 1.5^{21}, making 21 a reasonable guess for the greatest n with $1.5^n < 6051$.

To verify this guess, it suffices to verify that $1.5^{20} < 4034$. Expanding 1.5^{20} using the binomial theorem gives

$$\begin{aligned}
1.5^{20} &= (1.5^4)^5 \\
&= (5 + 0.0625)^5 \\
&= \binom{5}{0} \cdot 5^5 + \binom{5}{1} \cdot 5^4 \cdot 0.0625 + \binom{5}{2} \cdot 5^3 \cdot 0.0625^2 + \\
&\quad \binom{5}{3} \cdot 5^2 \cdot 0.0625^3 + \binom{5}{4} \cdot 5 \cdot 0.0625^4 + \binom{5}{5} \cdot 0.0625^5.
\end{aligned}$$

The largest binomial coefficient is $\binom{5}{2} = \binom{5}{3} = 10$, but $0.0625 < 0.1$, so when positive powers of 0.0625 are multiplied by binomial coefficients, the result is never more than 1. Replacing the binomial coefficients and powers of 0.0625 with 1 yields an upper bound of

$$1.5^{20} < 5^5 + 5^4 + 5^3 + 5^2 + 5^1 + 5^0 = \frac{5^6 - 1}{5 - 1} = 3906 < 4034.$$

Thus $1.5^{21} < 4034 \cdot 1.5 = 6051$, and so **21** is indeed the correct answer.

Problem 7. On July 17, 2017, the nation of Armlandia will turn n^2 years old and the nation of Nysmlistan will turn n years old. The next four anniversaries for which Nysmlistan's age divides Armlandia's age will occur, in order, on July 17 in the years 2027, 2032, 2038, and M. Compute M.

Solution 7. First note that the age of Nysmlistan in each of 2027, 2032, and 2038 is $n+10$, $n+15$, and $n+21$, respectively. Similarly, the age of Armlandia in those years is n^2+10, n^2+15, and n^2+21. Next suppose that $n+k \mid n^2+k$, for some integers n and k. Then using polynomial long division,

$$n^2 + k = (n-k)(n+k) + \frac{k(k+1)}{n+k}$$

which implies that $n+k$ must divide $k(k+1)$.

Therefore $n+10$ must divide 110, $n+15$ must divide 240, and $n+21$ must divide 462. Factoring each value shows that only two values of n satisfy these three conditions: $n=1$ and $n=45$. The condition that 2027 is the first year after 2017 in which Nysmlistan's age divides Armlandia's age precludes $n=1$, so $n=45$.

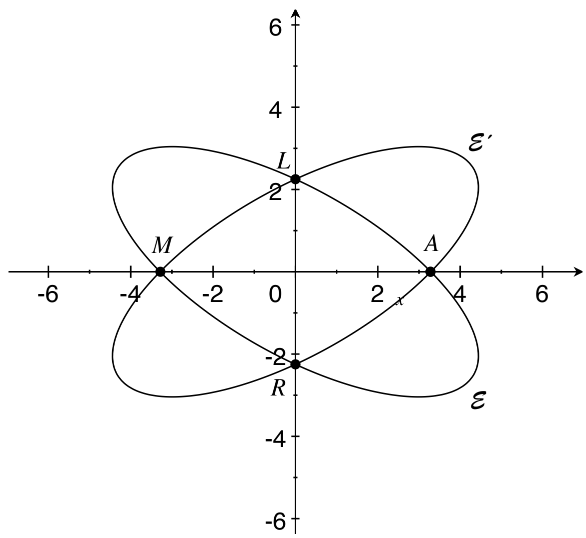
Let $m = M - 2017$, so that Nysmlistan's age in year M is $m+n$. Because $M > 2038$, $m+n > 21$. Then, following the above, $n+m \mid (n^2+m)$, so $n+m$ divides $m(m+1)$. Substituting $n=45$ gives $m+45 \mid (m^2+m)$, and using polynomial long division once again yields

$$m^2 + m = m - 44 + \frac{1980}{m+45}.$$

The first factors of 1980 greater than 45 are 55, 60, 66, and 90, corresponding to $m=10$, $m=15$, $m=21$, and $m=45$, respectively. Hence $m=45$ and $M = \mathbf{2062}$.

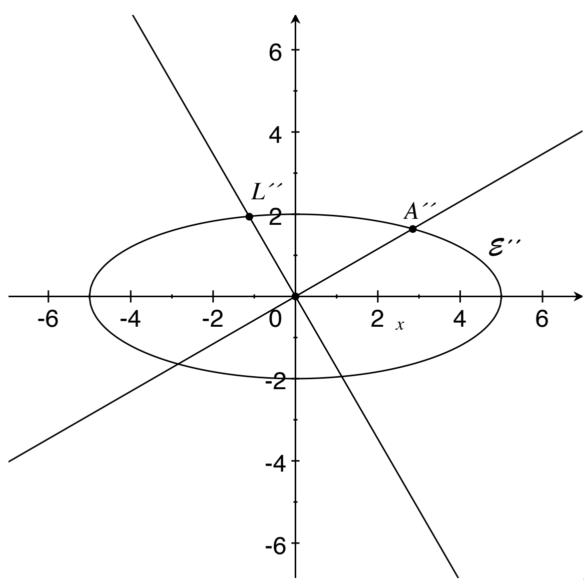
Problem 8. Ellipse $\mathcal{E}$ has center O, major axis of length 10, and minor axis of length 4. Ellipse $\mathcal{E}'$ is obtained by rotating $\mathcal{E}$ counterclockwise about O by 60°. Proceeding clockwise around the perimeter of $\mathcal{E}$, the intersection points of $\mathcal{E}$ and $\mathcal{E}'$ are labeled A, R, M, L. Compute $[AOL]$.

Solution 8. Place the ellipse and its images on the coordinate plane. While ordinarily it is more convenient to place ellipses so that their major and minor axes are parallel to (or coincide with) the x- and y-axes, in this situation, it's more strategic to place the ellipses so that their *intersection points* lie on the axes. Accordingly, place $\mathcal{E}$ on the xy-plane so that its center is at the origin and its major axis is rotated 30° clockwise relative to the x-axis. Then $\mathcal{E}'$ will also have its center at the origin, but will have its major axis rotated 30° counterclockwise relative to the x-axis. Because this diagram is symmetric about both $x-$ and $y-$ axes, the points of intersection A, R, M, and L all lie on the coordinate axes, as shown in the diagram below.



Without loss of generality, let $A = (a, 0)$ be the point of intersection on the positive x-axis. Then $L = (0, \ell)$ is the point of intersection on the positive y-axis, and $[AOL] = \frac{a\ell}{2}$.

To compute the lengths a and ℓ, rotate $\mathcal{E}$ by 30° counterclockwise to a new ellipse $\mathcal{E}''$ so that its major axis coincides with the x-axis. The points A and L will also be rotated by 30° to new points A'' and L'', respectively. Because $\triangle AOL \cong \triangle A''OL''$, it follows that $[AOL] = [A''OL''] = \frac{1}{2}A''O \cdot L''O$. Now A'' is the intersection of $\mathcal{E}''$ and a line through the origin making a 30° angle with the x-axis, as shown in the diagram below.



The equation for $\mathcal{E}''$ is $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$, or $4x^2 + 25y^2 = 100$. Because $\tan 30° = \frac{1}{\sqrt{3}}$, the equation for the line is

$y = \frac{1}{\sqrt{3}}x$. Substitute to obtain

$$\begin{aligned} 4x^2 + 25\left(\frac{1}{\sqrt{3}}x\right)^2 &= 100 \\ 12x^2 + 25x^2 &= 300 \\ x^2 &= \frac{300}{37} \\ x &= 10\sqrt{\frac{3}{37}}. \end{aligned}$$

Because $a = A''O$, it suffices to compute $A''O$ using trigonometry: $\frac{x}{A''O} = \cos 30^\circ = \frac{\sqrt{3}}{2}$, so $A''O = \frac{10\sqrt{\frac{3}{37}}}{\frac{\sqrt{3}}{2}} = \frac{20}{\sqrt{37}}$.

Similarly, L'' is the intersection of $\mathcal{E}''$ with the line $y = -\sqrt{3}x$. Solve for the intersection point to obtain

$$\begin{aligned} 4x^2 + 25\left(-\sqrt{3}x\right)^2 &= 100 \\ 79x^2 &= 100 \\ x &= -\frac{10}{\sqrt{79}}. \end{aligned}$$

Because $\ell = L''O$ and $\frac{x}{L''O} = \cos 120^\circ = -\frac{1}{2}$, conclude that $\ell = \frac{20}{\sqrt{79}}$.

Then $[AOL] = [A''OL''] = \dfrac{a\ell}{2} = \dfrac{\left(\frac{20}{\sqrt{37}}\right)\left(\frac{20}{\sqrt{79}}\right)}{2} = \dfrac{\mathbf{200\sqrt{2923}}}{\mathbf{2923}}$.

Problem 9. Let $E(n)$ denote the least integer strictly greater than n whose base-10 representation contains only even digits, and let $O(n)$ denote the least integer strictly greater than n whose base-10 representation contains only odd digits. Compute the least positive integer N for which $E(N) + O(N)$ is a multiple of 2017.

Solution 9. Let $S(n) = E(n)+O(n)$. In general, either $S(n)$ consists of only odd digits, or $S(n)$ has initial digit 2, and then only odd digits. To prove this observation, notice that one of $E(n)$ and $O(n)$ will consist of an initial digit, and then either all 0's or all 1's, respectively: if d is the first (leftmost) digit of n and d is odd, then the next larger number with all even digits must have first digit $d+1$ (or 2 if $d = 9$) followed by all 0's, while if d is even, then the next larger number with all odd digits must have first digit $d+1$ followed by all 1's. To compute the sum of $E(n)$ and $O(n)$, consider the sum of each pair of corresponding digits separately, beginning with the digits immediately to the right of their leftmost digits. If d is odd, this process is equivalent to adding 0's to $O(n)$'s digits. If d is even, this process is equivalent to adding 1's to $E(n)$'s digits, and none of $E(n)$'s digits is a 9. In neither case is there any need to carry. Thus when the $E(n)$ and $O(n)$ are added, there will be no carrying, except possibly in the initial digit. If $E(n)$ and $O(n)$ have the same number of digits, then the initial digit of $S(n)$ will be odd. If $E(n)$ and $O(n)$ do not have the same number of digits, then $O(n)$ must have initial digit 9 and $E(n)$ will have one more digit than $O(n)$, with an initial 2 and 0's everywhere else. Either way, $S(n)$ will fit the above form. This argument proves the lemma.

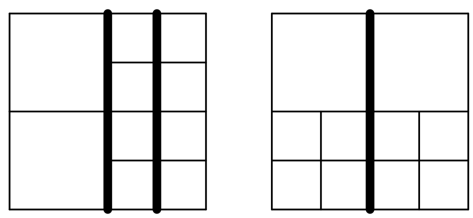
Therefore, to determine possible values of $S(N)$, look for multiples of 2017 that either consist of only odd digits or have an initial 2 and then all odd digits. Consider the thousands place of the first few multiples of 2017, which are $2017, 4034, 6051, \ldots$. The thousands place will continue to be even, until enough multiples of 17 are accumulated to roll over into the thousands place. As long as the thousands place is even, the multiple of 2017 cannot be a possible value of $S(N)$, because all the digits will not be even, and the thousands place is not the initial digit of the multiple (except for 2017, which does not work because 0 is not odd). Therefore, noting that $58 < 1000/17 < 59$, skip directly to $59 \cdot 2017 = 119003$. In order to force the hundreds place to be odd, note that $5 < 97/17 < 6$, so skip ahead by $6 \cdot 2017$ to $65 \cdot 2017 = 131105$. This value still doesn't work, so noting that $S(N)$ must be odd, consider $S(N) = 67 \cdot 2017 = 135139$.

Note that $S(N)$ does not begin with a 2, so $E(N)$ and $O(N)$ have the same number of digits, and the first digits of $E(N)$ and $O(N)$ must differ by 1, so $E(N)$ must begin with a 6 and $O(N)$ must equal 71111. Thus $E(N) = 135139 - 71111 = 64028$, and hence $N = \min(64028, 71111) - 2 = \mathbf{64026}$.

Problem 10. Compute the number of tilings of a 4×7 rectangle using only 1×1, 2×2, 3×3, and 4×4 tiles.

Solution 10. Let $T(n)$ be the number of ways to tile a $4 \times n$ rectangle with the given types of tiles, with $T(0) = 1$. Because there is only one way to tile a 4×1 rectangle, i.e., with 1×1 squares, $T(1) = 1$.

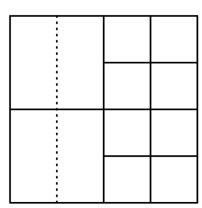
Use the term *fault line* to describe a vertical line from top to bottom that does not intersect any tiles and that is not either the left or the right border of the figure. For example, if a 4×4 region is tiled with two 2×2 tiles on the left, and then eight 1×1 tiles, as shown in the left figure below, then the tiling has two fault lines, while if the two 2×2 tiles are next to each other at the top of the region, then the tiling has only one fault line, as shown in the right figure below.



Two fault lines One fault line

Under this definition, it is possible for a tiling to have no fault lines; for example, if the region is completely filled by a single square, or if 2×2 squares are staggered so that the right half of one is directly above or below the left half of the other. Thus, in general, a $4 \times n$ rectangle can have between 0 and $n - 1$ fault lines, inclusive. The number of tilings can be counted by conditioning on the location of the leftmost fault line.

For ease of notation, label the vertical lines (whether or not they are fault lines) one unit to the right of the leftmost edge, two units to the right of the leftmost edge, etc. as ℓ_1, ℓ_2, etc. respectively, as shown below. In that example, the leftmost fault line is at ℓ_2.

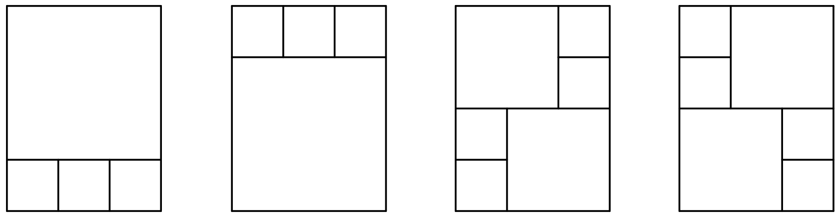


ℓ_1 ℓ_2 ℓ_3 ℓ_4

Let $F(n)$ represent the number of ways to tile a $4 \times n$ rectangle with *no* fault lines. Note that $F(0) = T(0) = 1$ and $F(1) = T(1) = 1$.

A 4×2 rectangle can be tiled with no fault line in four ways (two 2×2 squares or one 2×2 square and four 1×1 squares), so $F(2) = 4$. If there is one fault line, it is at ℓ_1, and divides the rectangle into two 4×1 strips, each of which can be tiled in $F(1) = T(1)$ ways, for $F(1) \cdot T(1) = 1$ tiling. So $T(2) = F(2) + F(1) \cdot T(1) = 5$.

A 4×3 rectangle can be tiled with no fault line in four ways: either there is a 3×3 square (and the rest of the area is filled in with 1×1 squares), which can happen in two ways, or there are two 2×2 squares, one above and to either the left or right of the other, with the other spaces filled in with 1×1 squares.



Hence $F(3) = 4$. If the leftmost fault line is at ℓ_1, then there are $F(1)$ ways to tile the left side and $T(2)$ ways to tile the right side. If the leftmost fault line is at ℓ_2, then there are $F(2)$ ways to tile the left side and $T(1)$ ways to tile the right side. Hence $T(3) = F(3) + F(1) \cdot T(2) + F(2) \cdot T(1) = 4 + 1 \cdot 5 + 4 \cdot 1 = 13$.

A 4×4 rectangle can be tiled with no fault line in three ways: either using a single 4×4 square, or by alternating 2×2 squares in the top and bottom row (and filling in with four 1×1 squares on the ends) as in the 4×3 case, which can happen in two ways. So $F(4) = 3$. Otherwise, with the first fault line occurring at ℓ_1, ℓ_2, or ℓ_3, there are $F(1) \cdot T(3)$, $F(2) \cdot T(2)$, and $F(3) \cdot T(1)$ possible tilings, respectively, so that

$$\begin{aligned} T(4) &= F(4) + F(1) \cdot T(3) + F(2) \cdot T(2) + F(3) \cdot T(1) \\ &= 3 + 1 \cdot 13 + 4 \cdot 5 + 4 \cdot 1 \\ \implies T(4) &= 40. \end{aligned}$$

The foregoing analysis shows that, in general,

$$T(n) = F(n) + F(1) \cdot T(n-1) + F(2) \cdot T(n-2) + \cdots + F(n-1) \cdot T(1).$$

Indeed, the proof is trivial, because if there is no fault line, there are $F(n)$ tilings, and if the leftmost fault line is at ℓ_i, then there are $F(i)$ ways of tiling the region to the left of the fault line without any fault lines, and there are $T(n-i)$ ways of tiling the region to the right of the fault line (with or without fault lines). Also, for $n > 4$, $F(n) = 2$, because the only way to tile such a region without fault lines is to alternate 2×2 squares in the top and bottom rows. So a simpler version of the formula above for $n > 4$ is simply

$$T(n) = 1 \cdot T(n-1) + 4 \cdot T(n-2) + 4 \cdot T(n-3) + 3 \cdot T(n-4) + 2 \cdot T(n-5) + \cdots + 2 \cdot T(0).$$

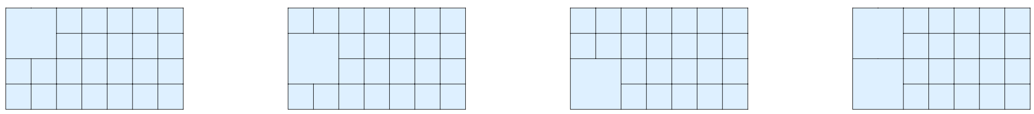
Using this formula, construct a table of values, as shown below.

n	0	1	2	3	4	5	6	7
$T(n)$	1	1	5	13	40	117	348	**1029**

Alternate Solution: Let $T(n)$ be the number of ways to tile an $4 \times n$ rectangle with square integer-length tiles, and let $S(n, k)$ be the number of ways to tile the same rectangle conditional on the largest tile touching the left-hand border being of size $k \times k$. Then $T(n) = S(n, 1) + S(n, 2) + S(n, 3) + S(n, 4)$.

Because there is only one way to arrange a strip of 1×1 squares along the left side, $S(n, 1) = T(n-1)$. Because there is only one way to arrange a single 4×4 square on the left side, $S(n, 4) = T(n-4)$. As the diagram below shows, there are two ways to arrange a 3×3 square and three 1×1 squares so that the 3×3 square touches the left side, so $S(n, 3) = 2 \cdot T(n-3)$.

The remaining term to address is the $S(n,2)$ term. There are three ways to fit one 2×2 tile and four 1×1 tiles in the leftmost 4×2 subregion. There is also one way to fit two 2×2 tiles in that subregion, as illustrated below.



However, it is also possible for a series of two or more 2×2 squares to alternate across an $4\times m$ range, where m can be any integer larger than 2. Note that for any such m, this alternating set of 2×2 squares can occur in two ways, depending on whether the leftmost 2×2 square is at the top or the bottom. The diagram below shows both examples with $m=4$.

Thus $S(n,2) = 4\cdot T(n-2) + 2\cdot \sum_{m\geq 3} T(n-m)$.

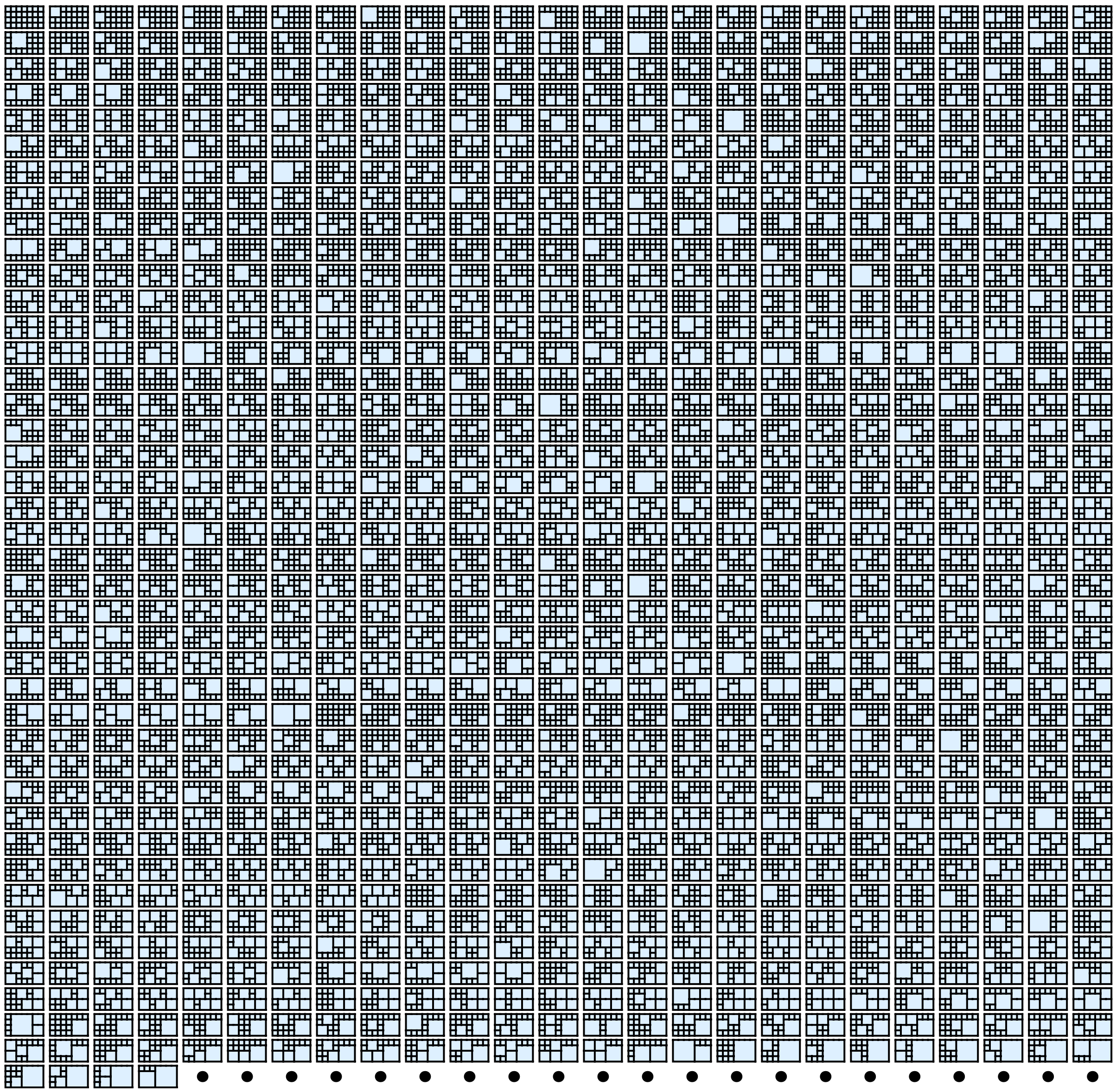
Combine the above results to obtain the following recursion:

$$T(n) = T(n-1) + 4\cdot T(n-2) + 2\cdot T(n-3) + T(n-4) + 2\cdot \sum_{m\geq 3} T(n-m).$$

Use the recursion to build the table below.

n	0	1	2	3	4	5	6	7
$T(n)$	1	1	5	13	40	117	348	1029

Hence there are **1029** tilings. They are illustrated on the following page.



4 Power Question 2017: The Paral-Elle Universe

Instructions: The power question is worth 50 points; each part's point value is given in brackets next to the part. To receive full credit, the presentation must be legible, orderly, clear, and concise. If a problem says "list" or "compute," you need not justify your answer. If a problem says "determine," "find," or "show," then you must show your work or explain your reasoning to receive full credit, although such explanations do not have to be lengthy. If a problem says "justify" or "prove," then you must prove your answer rigorously. Even if not proved, earlier numbered items may be used in solutions to later numbered items, but not vice versa. Pages submitted for credit should be NUMBERED IN CONSECUTIVE ORDER AT THE TOP OF EACH PAGE in what your team considers to be proper sequential order. PLEASE WRITE ON ONLY ONE SIDE OF THE ANSWER PAPERS. Put the TEAM NUMBER (not the team name) on the cover sheet used as the first page of the papers submitted. Do not identify the team in any other way.

Elle is a student at Springfield High School. One day, she's escorted from her math class by a friendly space alien, who takes her to a parallel universe to learn math the way they do math, which is different than the way we do math. In the parallel universe, students are taught two operations: $\oplus$ and $\odot$. Elle notices that

$$a \oplus b = \min(a, b) \quad \text{and} \quad a \odot b = a + b$$

for all real numbers a and b. Elle decides to investigate the properties of these two new operations.

Throughout this Power Question, $a \oplus b$ and $a \odot b$ will always denote parallel-universe addition and multiplication, respectively, while $a + b$ and $a \cdot b$ will denote addition and multiplication in the usual sense. The standard order of operations ($\odot$ first, then $\oplus$) will apply, although parentheses are sometimes included for clarification. All variables refer to real numbers unless otherwise specified.

1. **a.** Determine whether there is a real number $\mathcal{O}$ such that $\mathcal{O} \oplus x = x$ for all x. [2 pts]

   b. Show that there is a real number $\mathcal{I}$ such that $\mathcal{I} \odot x = x$ for all x. [2 pts]

2. Show that $\odot$ distributes over $\oplus$; that is, $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ for all a, b, c. [3 pts]

3. Find all solutions to each of the following equations:

   a. $(x \odot x) \oplus (2 \odot x) \oplus 1 = x \oplus 2$; [3 pts]

   b. $(x \odot x) \oplus (-1) = (x \oplus 1) \odot (x \oplus (-1))$. [3 pts]

Next, Elle investigates the properties of exponents and polynomials. She uses the notation

$$a^{\odot b} = \underbrace{a \odot a \odot a \odot \cdots \odot a}_{b \text{ copies of } a},$$

where a is a real number and b is a positive integer. Furthermore, by definition, $a^{\odot 0} = 0$. Elle wonders if exponentiation works in the parallel universe the way it works in ours.

4. Show that for any real numbers x and y, and any positive integer n:

   a. $(x \odot y)^{\odot n} = x^{\odot n} \odot y^{\odot n}$; [2 pts]

   b. $(x \oplus y)^{\odot n} = x^{\odot n} \oplus y^{\odot n}$; [2 pts]

   c. $(x \oplus y)^{\odot 2} = x^{\odot 2} \oplus y^{\odot 2} = (x^{\odot 2}) \oplus (x \odot y) \oplus (y^{\odot 2})$. [3 pts]

5. **a.** Show that for any a, there is exactly one solution to the equation $x^{\odot 2} = a$. This is called the **parallel square root of a** and is denoted by $\sqrt[\circ]{a}$. [2 pts]

   b. Show that for any a, there is exactly one solution to the equation $a \odot x = \mathcal{I}$, where $\mathcal{I}$ is defined in Problem 1b. This is called the **parallel reciprocal of a** and is denoted by $\widehat{a}$. [2 pts]

   c. Show that for any a, $\sqrt[\circ]{\widehat{a}} = \widehat{\sqrt[\circ]{a}}$. [3 pts]

In the parallel universe, a **parallel-universe polynomial (PUP)** of degree n is a finite "sum" (i.e., parallel-universe addition) of $n+1$ parallel products whose exponents decrease from n to 0, inclusive:

$$f(x) = (a_n \odot x^{\odot n}) \oplus (a_{n-1} \odot x^{\odot(n-1)}) \oplus \cdots \oplus (a_1 \odot x^{\odot 1}) \oplus (a_0 \odot x^{\odot 0}),$$

where each a_i is a real number. As a convenient shorthand, $x^{\odot 1}$ will simply be written as x and $a_0 \odot x^{\odot 0}$ will simply be written as a_0. For example:

- $f(x) = (5 \odot x) \oplus 3$ is a degree-1 PUP, but not a degree-2 PUP.
- $f(x) = (1 \odot x^{\odot 2}) \oplus 1$ is not a PUP at all, because the x term is missing.
- $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 0$ is a degree-2 PUP.
- $f(x) = (x^{\odot 2}) \oplus x \oplus 1$ is not a PUP; it should instead be written as $(0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 1$.

Elle now examines the properties of degree-2 (quadratic) PUPs. In the parallel universe, a **root** of a PUP f is a real number x such that $f(x) = 0$.

6. In our universe, a quadratic equation has either zero, one, or two distinct real roots.

a. Find a quadratic PUP that has exactly one root. [2 pts]

b. Could a quadratic PUP have zero roots? Justify your answer. [2 pts]

c. Could a quadratic PUP have infinitely many distinct roots? Justify your answer. [2 pts]

d. Suppose that $f(x)$ is a PUP of positive degree. Prove that if f has more than one root, then it has infinitely many distinct roots. [3 pts]

7. Suppose $f(x) = (a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ is a quadratic PUP that has exactly one root. Find a formula for this root in terms of a, b, and c. Express the formula using only the operations $\oplus$, $\odot$, parallel-universe exponentiation, parallel square roots, and parallel reciprocals. [5 pts]

Suppose $f(x) = (a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ is a quadratic PUP. A **factorization** of f is an ordered triple of real numbers (k, r, s) such that $r \le s$, and

$$f(x) = k \odot (x \oplus r) \odot (x \oplus s),$$

for all real numbers x.

8. Prove that if (k, r, s) is a factorization of f, then:

a. $k = a$, and [2 pts]

b. $r \odot s = c \odot \widehat{a}$. [2 pts]

9. Prove that every quadratic PUP has exactly one factorization. [5 pts]

5 Solutions to Power Question

1. **a.** No. This equation would mean that $\min(\mathcal{O}, x) = x$ for all x. However, if $\mathcal{O}$ exists, then $\min(\mathcal{O}, \mathcal{O}+1) = \mathcal{O} \neq \mathcal{O}+1$. This is a contradiction, so no such $\mathcal{O}$ exists.

b. $\mathcal{I} = 0$. This makes $\mathcal{I} \odot x = \mathcal{I} + x = 0 + x = x$ for all real numbers x.

2. Translating this into "normal" arithmetic, the left-hand side is $a + \min(b, c)$, and the right-hand side is $\min(a+b, a+c)$. If $b \leq c$, then both sides are equal to $a+b$, and if $b > c$, then both sides are equal to $a+c$. Thus this equation holds for all real numbers a, b, c.

3. **a.** The equation translates to $\min(2x, x+2, 1) = \min(x, 2)$. If $x < \dfrac{1}{2}$, the equation is equivalent to $2x = x$, which implies $x = 0$. If $x \geq \dfrac{1}{2}$, the equation is equivalent to $1 = \min(x, 2)$, which implies $x = 1$. These both satisfy the original equation, and there cannot be any other solutions. Thus the two solutions are $x = 0$ and $x = 1$.

b. This equation translates to $\min(2x, -1) = \min(x, 1) + \min(x, -1)$. If $x \leq -1$, then both sides are equal to $2x$, so every $x \leq -1$ is a solution. If $-1 < x < -\frac{1}{2}$, then the equation becomes $2x = x - 1$, which has no solution on the interval. If $-\frac{1}{2} \leq x \leq 1$, then the equation is $-1 = x - 1$, so $x = 0$ is a solution. Finally, if $x > 1$, then the equation becomes $-1 = 0$, which is not true for any value of x. Thus the solutions to this equation are $x = 0$ and all $x \leq -1$.

4. Note that, translating into normal arithmetic, $a^{\odot b} = b \cdot a$. Each of the statements will be proven by translation.

a. This equation translates to $n(x+y) = nx + ny$, which is just the normal distributive property of multiplication over addition.

b. This translates to $n \cdot \min(x, y) = \min(nx, ny)$. If $x \leq y$, then both sides are equal to nx, and if $x > y$, then both sides are equal to ny. Either way, the equation holds for all real x and y and all integers n.

c. The first equality holds by part (b). For the second, the left-hand side is $\min(2x, 2y)$, and the right-hand side is $\min(2x, x+y, 2y)$. But note that $x+y$ is the arithmetic mean of $2x$ and $2y$, so it must be greater than or equal to one of them and less than or equal to the other. Thus $\min(2x, x+y, 2y) = \min(2x, 2y)$, as desired.

5. **a.** This equation translates to $2x = a$, which has exactly one solution: $x = \sqrt[\circ]{a} = \frac{a}{2}$.

b. This equation translates to $a + x = \mathcal{I} = 0$, which has exactly one solution: $x = \widehat{a} = -a$.

c. As shown in parts (a) and (b), the left-hand side is equal to $\frac{1}{2}(-a)$, and the right-hand side is equal to $-\left(\frac{1}{2}a\right)$, and these are equal by associativity of (normal) multiplication.

6. **a.** Let $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 1$. Then $f(x) = \min(2x, x, 1)$, which is $2x$ for $x \leq 0$, x for $0 < x < 1$, and 1 for $x \geq 1$. The only root of this PUP is $x = 0$.

b. The answer is yes. As an example, let $f(x) = (4 \odot x^{\odot 2}) \oplus (3 \odot x) \oplus (-1)$. Then $f(x) = \min(2x+4, x+3, -1)$, so $f(x) \leq -1$ for all x, which means f has no roots.

c. The answer is yes. As an example, let $f(x) = (0 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus 0$. Then $f(x) = \min(2x, x, 0) = 0$ for all $x \geq 0$, so all nonnegative real numbers are roots of f.

Note: A general quadratic PUP $(a \odot x^{\odot 2}) \oplus (b \odot x) \oplus c$ has no roots if $c < 0$, infinitely many roots if $c = 0$, and one root if $c > 0$. A proof of this fact is in the solution to Problem 7.

d. Consider a general PUP, $f(x)$, as defined in the background to Problem 6. Then

$$f(x) = \min(nx + a_n, (n-1)x + a_{n-1}, \ldots, 2x + a_2, x + a_1, a_0).$$

This is the minimum of $n+1$ functions, all of whose graphs are lines of nonnegative slope. Suppose $x \leq y$. Then $kx + a_k \leq ky + a_k$ for each $k \geq 0$, and therefore $f(x) \leq f(y)$. Now suppose f has two different roots, so there are real numbers r_1, r_2 with $r_1 < r_2$ and $f(r_1) = f(r_2) = 0$. Now, for any z with $r_1 < z < r_2$, $f(r_1) \leq f(z) \leq f(r_2)$, so it follows that $f(z) = 0$. Thus every z in the interval $[r_1, r_2]$ is a root of f, hence f has infinitely many roots. As seen in parts (a) and (b), if f does not have infinitely many roots, then it can only have either one root or zero roots. This shows that f has either zero, one, or infinitely many roots.

7. *There are multiple ways to solve this problem; in particular, the root can be found by translating to normal arithmetic. However, a solution using almost exclusively parallel-universe arithmetic follows.*

Setting $f(x) = 0$ and translating to normal arithmetic, note that the given equation is equivalent to $f(x) = \min(2x + a, x + b, c) = 0$. First, note that if $x > \max\left(\frac{c-a}{2}, c - b\right)$, then $f(x) = c$, and $f(x) \leq c$ for all x. Thus if $c < 0$, then f has no roots, and if $c = 0$, then f has infinitely many roots. Hence if f has exactly one root, it follows that $c > 0$. Notice that in general, $\widehat{(a^{\odot n})} = (\widehat{a})^{\odot n}$. This is because this equation translates to $-(na) = n(-a)$, which holds by associativity of (normal) multiplication. Now let $x = \widehat{\sqrt[\circ]{a} \oplus b}$; this is the root of f. Substituting and using the result from the above paragraph yields

$$f(x) = a \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}} \oplus b \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)} \oplus c.$$

Form the parallel-universe equivalent of a common denominator:

$$f(x) = \left(a \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}\right) \oplus \left(b \odot \left(\sqrt[\circ]{a} \oplus b\right) \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}\right) \oplus \left(c \odot \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2} \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}\right).$$

Now use the distributive property, effectively combining like terms:

$$f(x) = \left[a \oplus b \odot \left(\sqrt[\circ]{a} \oplus b\right) \oplus c \odot \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}\right] \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}.$$

Using the results of Problems 2 and 4 yields:

$$f(x) = \left[a \oplus \left(b \odot \sqrt[\circ]{a}\right) \oplus b^{\odot 2} \oplus (c \odot a) \oplus \left(c \odot b^{\odot 2}\right)\right] \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}.$$

Now recall the assumption that $c > 0$, and therefore $a \oplus (c \odot a) = a$, and $b^{\odot 2} \oplus \left(c \odot b^{\odot 2}\right) = b^{\odot 2}$. Thus

$$f(x) = \left[a \oplus \left(b \odot \sqrt[\circ]{a}\right) \oplus b^{\odot 2}\right] \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}}.$$

By Problem 4c, this becomes:

$$f(x) = \left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2} \odot \widehat{\left(\sqrt[\circ]{a} \oplus b\right)^{\odot 2}} = 0,$$

as desired.

Other forms of the answer, such as $\sqrt[\circ]{\widehat{(a \oplus b^{\odot 2})}}$*, are also possible. However, answers like* $\sqrt[\circ]{\widehat{a}} \oplus \widehat{b}$ *are **not** valid, because Problem 4a does not hold if n is a negative integer.*

8. Before proving any part of this problem, let (k, r, s) be a factorization of f, and let

$$g(x) = k \odot (x \oplus r) \odot (x \oplus s),$$

so that $f(x) = g(x)$ for all x. By the result of Problem 2,

$$g(x) = (k \odot x^{\odot 2}) \oplus (k \odot (r \oplus s) \odot x) \oplus (k \odot r \odot s).$$

Also, note as in Problem 7 that $f(x) = \min(2x + a, x + b, c)$.

a. If $x < \min\left(\frac{c-a}{2}, b - a\right)$, then $f(x) = 2x + a = a \odot x^{\odot 2}$. Similarly, if $x < r$, then $g(x) = k \odot x^{\odot 2}$. Choose a value of x small enough to satisfy both inequalities. Because $f(x) = g(x)$, it follows that $a \odot x^{\odot 2} = k \odot x^{\odot 2}$. Multiplying (in the parallel universe) by $\widehat{(x^{\odot 2})}$ yields $k = a$, as desired.

b. As argued in Problem 7, if $x > \max\left(\frac{c-a}{2}, c - b\right)$, then $f(x) = c$. Similarly, if $x > s$, then $g(x) = k \odot r \odot s$. Choose x large enough to satisfy both inequalities. Because $f(x) = g(x)$, it follows that $c = k \odot r \odot s$. By part (a), $k = a$, and then by the definition of the parallel reciprocal, $r \odot s = c \odot \widehat{a}$.

9. As in Problem 8, let (k, r, s) be a factorization of a quadratic PUP $f(x)$, and let $g(x) = k \odot (x \oplus r) \odot (x \oplus s)$. Note by the distributive property (Problem 2), that

$$g(x) = (k \odot x^{\odot 2}) \oplus (k \odot r \odot x) \oplus (k \odot r \odot s)$$

because $r \leq s$ and thus $r \oplus s = r$. Furthermore, from Problems 8a and 8b, this becomes

$$g(x) = (a \odot x^{\odot 2}) \oplus (a \odot r \odot x) \oplus c.$$

Claim: If $2b < a + c$, then the only factorization of f is $a \odot \big(x \oplus (b - a)\big) \odot \big(x \oplus (c - b)\big)$ (where "$-$" denotes normal subtraction), and if $2b \geq a + c$, then the only factorization of f is $a \odot (x \oplus d) \odot (x \oplus d)$, where $d = \frac{c-a}{2}$.

To prove the claim, consider each case of the claim separately:

Case 1: Assume $2b < a + c$, so that $b - a < c - b$. Let $x = \frac{c-a}{2}$. Then

$$f(x) = \min(2x + a, x + b, c) = \min\left(c, \frac{c-a}{2} + b, c\right).$$

Because $2b < a + c$, it follows that $c - a < 2c - 2b$, and thus $x < c - b$. Then $x + b < c$, so $f(x) = x + b$.

Note also that

$$g(x) = \min(2x + a, x + a + r, c) = \min\left(c, \frac{c+a}{2} + r, c\right).$$

Because $f(x) = x + b < c$, conclude that $g(x) \neq c$, hence $g(x) = \frac{c+a}{2} + r$. Setting this equal to $f(x)$ yields $r = b - a$, and the result of Problem 8b gives $s = c - b$.

This is the only possible factorization of f, and it remains to show that it is a factorization. Recall that

$$g(x) = (a \odot x^{\odot 2}) \oplus (a \odot r \odot x) \oplus (a \odot r \odot s).$$

Using the definition of $\odot$, $a \odot r = b$ and $a \odot r \odot s = c$, so $g(x) = f(x)$ for all x, as desired.

Case 2: Now assume $2b \geq a + c$, so that $c - b \leq b - a$. First, note that for every x, either $x \leq b - a$ (and then $2x + a \leq x + b$), or $c - b \leq x$ (and then $c \leq x + b$). Thus

$$f(x) = \min(2x + a, x + b, c) = \min(2x + a, c).$$

Now consider the claim that $r \geq c - a - r$. If not, then there exists an x with $r < x < c - a - r$. Because $r < x$, it follows that $x + a + r < 2x + a$. Note that $x < c - a - r$ is equivalent to $x + a + r < c$. This implies that $g(x) = x + a + r$. And because $f(x)$ is equal to either $2x + a$ or c, it follows that $x = r$ or $x = c - a - r$, a contradiction. Therefore $r \geq c - a - r$, and rearranging, $r \geq \frac{c-a}{2}$.

Now consider s. By Problem 8b, $r + s = c - a$, and $r \leq s$ by assumption, so $2r \leq c - a$, and $r \leq \frac{c-a}{2}$. Thus the only possible factorization of f has $r = s = \frac{c-a}{2}$ (and $k = a$).

It remains to verify that this is a factorization of f. For convenience, let $d = \frac{c-a}{2}$. An earlier result showed that

$$f(x) = \min(2x + a, c) = (a \odot x^{\odot 2}) \oplus c.$$

Now apply the result of Problem 4b to g:

$$g(x) = a \odot (x \oplus d)^{\odot 2} = (a \odot x^{\odot 2}) \oplus (a \odot d^{\odot 2}) = (a \odot x^{\odot 2}) \oplus c = f(x),$$

as desired.

Authors' Note: Elle has in fact discovered the relatively new field of *tropical algebra*, which is currently an area of active mathematical research. For a further introduction to the field, including applications to algebraic geometry and computational biology, see the article "Tropical Mathematics" by David Speyer and Bernd Sturmfels, appearing in the June 2009 issue of *Mathematics Magazine*. This article was also invaluable to the authors as a source of inspiration and a few problems in this Power Question. It is available online at `https://math.berkeley.edu/~bernd/mathmag.pdf`.

6 Individual Problems

Problem 1. Compute the number of perfect squares in the set $\{1^1, 2^2, 3^3, \ldots, 2017^{2017}\}$.

Problem 2. Trapezoid $ARML$ has $\overline{AR} \parallel \overline{ML}$. Given that $AR = 4$, $RM = \sqrt{26}$, $ML = 12$, and $LA = \sqrt{42}$, compute AM.

Problem 3. Compute the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common.

Problem 4. In $\triangle ABC$, $\mathrm{m}\angle A = 90^\circ$, $AC = 1$, and $AB = 5$. Point D lies on ray $\overrightarrow{AC}$ such that $\mathrm{m}\angle DBC = 2\mathrm{m}\angle CBA$. Compute AD.

Problem 5. Given that $2^{-\frac{3}{2}+2\cos\theta} + 1 = 2^{\frac{1}{4}+\cos\theta}$, compute $\cos 2\theta$.

Problem 6. A diagonal of a regular 2017-gon is chosen at random. Compute the probability that the chosen diagonal is longer than the median length of all of the diagonals.

Problem 7. Given that $i = \sqrt{-1}$, compute $(i+1)^3(i-2)^3 + 3(i+1)^2(i+3)(i-2)^2 + 3(i+1)(i+3)^2(i-2) + (i+1)^3$.

Problem 8. In triangle ABC, $\mathrm{m}\angle C = 90^\circ$ and $BC = 17$. Point E lies on side $\overline{BC}$ such that $\mathrm{m}\angle CAE = \mathrm{m}\angle EAB$. The circumcircle of triangle ABE passes through a point F on side $\overline{AC}$. Given that $CF = 3$, compute AB.

Problem 9. Let S be the set of divisors of $67 \cdot 9! + 27 \cdot 8!$. Compute the median of S.

Problem 10. Rhombus $ARML$ has its vertices on the graph of $y = \lfloor x \rfloor - \{x\}$. Given that $[ARML] = 8$, compute the least upper bound for $\tan A$.

7 Answers to Individual Problems

Answer 1. 1030

Answer 2. $\sqrt{66}$

Answer 3. 12

Answer 4. $\frac{37}{11}$ $\left(\text{or } 3\frac{4}{11}\right)$ or $3.\overline{36}$

Answer 5. $\frac{1}{8}$ (or 0.125)

Answer 6. $\frac{503}{1007}$

Answer 7. $-24i - 20$ or $-20 - 24i$

Answer 8. $\frac{149}{3}$ $\left(\text{or } 49\frac{2}{3} \text{ or } 49.\overline{6}\right)$

Answer 9. 5040

Answer 10. $\frac{100}{621}$

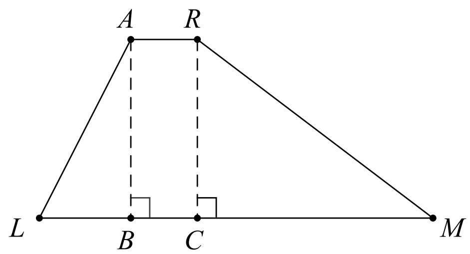
8 Solutions to Individual Problems

Problem 1. Compute the number of perfect squares in the set $\{1^1, 2^2, 3^3, \ldots, 2017^{2017}\}$.

Solution 1. For a positive integer k, consider whether k^k is even or odd. If k^k is even, then k is also even, and $k^k = (k^{k/2})^2$ is a perfect square. There are $\lfloor 2017/2 \rfloor = 1008$ even k^k in the set. If k^k is odd, then k is also odd, and k^k will be a perfect square if and only if k is a perfect square. Because $44^2 < 2017 < 45^2$, and 22 of the first 44 natural numbers are odd, there are 22 odd squares in the set. The total number of perfect squares in the given set is therefore $1008 + 22 = \mathbf{1030}$.

Problem 2. Trapezoid $ARML$ has $\overline{AR} \parallel \overline{ML}$. Given that $AR = 4$, $RM = \sqrt{26}$, $ML = 12$, and $LA = \sqrt{42}$, compute AM.

Solution 2. Draw lines perpendicular to $\overline{LM}$ through A and R, meeting $\overline{LM}$ at B and C, respectively, as shown in the diagram below. (As discussed in the note below, it can be shown that B and C must lie on the segment $\overline{LM}$.)



By the Pythagorean Theorem, $AB^2+LB^2 = 42$ and $RC^2+CM^2 = 26$. Because $AB = RC$, $LB+BC+CM = 12$, and $RC^2 + CM^2 = 26$, it follows that

$$AB^2 + (8 - LB)^2 = 26$$
$$\implies AB^2 + 64 - 16LB + LB^2 = 26.$$

Substitute 42 for $AB^2 + LB^2$ to obtain $64 - 16LB + 42 = 26$, from which it follows that $LB = 5$, and $AB = \sqrt{42 - 5^2} = \sqrt{17}$. Apply the Pythagorean Theorem once more to obtain

$$\begin{aligned} AM &= \sqrt{AB^2 + BM^2} \\ &= \sqrt{AB^2 + (LM - LB)^2} \\ &= \sqrt{AB^2 + (12 - 5)^2} \\ &= \sqrt{\mathbf{66}}. \end{aligned}$$

Note: In general, trapezoids with a given set of side lengths are unique, i.e., there is a SSSS congruence theorem for trapezoids. In this case, assuming that C does not lie on $\overline{LM}$ leads to a contradiction as follows. Let $CM = x$ and $RC = h$, implying $BL = (x + 12) - 4 = x + 8$. Then $x^2 + h^2 = 26$ and $(x + 8)^2 + h^2 = 42$, yielding $x = -3$, but side lengths must be positive, so this is a contradiction.

Problem 3. Compute the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common.

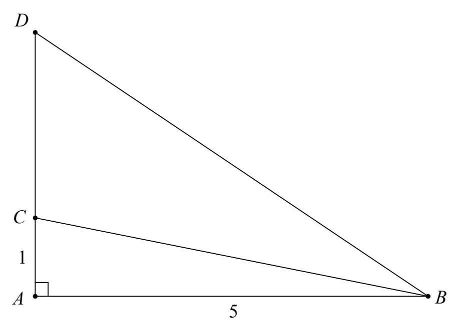
Solution 3. In general, if r is a root of $f(x)$ and of $g(x)$, then r is a root of $f(x) - g(x)$. The common root of $x^2 - ax + 24$ and $x^2 - bx + 36$ is a root of $(b-a)x - 12$. It is evident that there are no solutions where $b = a$, because the constant polynomial $0x - 12$ has no roots. It follows that the common root of the two polynomials is $\dfrac{12}{b-a}$. Substitute this root into $x^2 - ax + 24$, and notice that

$$\begin{aligned} 0 &= \frac{144}{(b-a)^2} - a \cdot \frac{12}{b-a} + 24 \\ 0 &= 144 - 12a(b-a) + 24(b-a)^2 \\ -12 &= 3a^2 + 2b^2 - 5ab \\ -12 &= (a-b)(3a-2b). \end{aligned}$$

Because $a - b$ and $3a - 2b$ are integers, $a - b$ must be an integer that divides 12. If k divides 12, then $a - b = k$ implies that $a = k + b$, so $3(k+b) - 2b = -\frac{12}{k}$. Solve for b to obtain $b = -\frac{12}{k} - 3k$, which is also an integer. Thus choosing any integer that divides 12 for $a - b$ will result in a distinct ordered pair of integers (a, b). There are 12 integers that divide 12 (namely ± 1, ± 2, ± 3, ± 4, ± 6, and ± 12), so there are **12** such ordered pairs of integers.

Problem 4. In $\triangle ABC$, $\mathrm{m}\angle A = 90°$, $AC = 1$, and $AB = 5$. Point D lies on ray $\overrightarrow{AC}$ such that $\mathrm{m}\angle DBC = 2\mathrm{m}\angle CBA$. Compute AD.

Solution 4. Consider the diagram below.



Because $\mathrm{m}\angle DBC = 2\mathrm{m}\angle CBA$, it follows that $\mathrm{m}\angle DBA = 3\mathrm{m}\angle CBA$. Thus $\dfrac{AD}{AB} = \dfrac{AD}{5} = \tan(3\mathrm{m}\angle CBA)$. Using the triple-angle formula, $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$, substitute to obtain $\dfrac{AD}{5} = \dfrac{3 \cdot \frac{1}{5} - \frac{1}{125}}{1 - 3 \cdot \frac{1}{25}}$, so $AD = \dfrac{3 - \frac{1}{25}}{1 - \frac{3}{25}} = \dfrac{75-1}{25-3} = \mathbf{\dfrac{37}{11}}$.

Problem 5. Given that $2^{-\frac{3}{2} + 2\cos\theta} + 1 = 2^{\frac{1}{4} + \cos\theta}$, compute $\cos 2\theta$.

Solution 5. Let $x = 2^{\cos\theta}$ and let $k = 2^{-3/2}$. Then $\frac{1}{k} = \sqrt{8} \Rightarrow \frac{1}{2k} = \sqrt{2} \Rightarrow \sqrt{\frac{1}{2k}} = \sqrt[4]{2}$. Then the given equation becomes $kx^2 + 1 = x\sqrt{\frac{1}{2k}}$, which is equivalent to $x^2 - \frac{x}{k}\sqrt{\frac{1}{2k}} + \frac{1}{k} = x^2 - x \cdot \sqrt{8} \cdot \sqrt{\frac{1}{2k}} + \frac{1}{k} = x^2 - 2x\sqrt{\frac{1}{k}} + \frac{1}{k} = \left(x - \sqrt{\frac{1}{k}}\right)^2 = 0$. Thus $2^{\cos\theta} = x = \sqrt{\frac{1}{k}} = 2^{3/4}$ and therefore $\cos\theta = \frac{3}{4}$. Hence $\cos 2\theta = 2\cos^2\theta - 1 = 2 \cdot \left(\frac{3}{4}\right)^2 - 1 = \mathbf{\frac{1}{8}}$.

Alternate Solution: The expressions in the exponents $\cos\theta$ and $2\cos\theta$ are suggestive of a quadratic equation. Let $x = 2^{\frac{1}{4}+\cos\theta}$. Then $x^2 = 2^{\frac{1}{2}+2\cos\theta}$; note that this exponent is 2 more than the exponent on the left-hand side. Because $2^2 = 4$, rewrite the left side as $\frac{1}{4}x^2+1$ to obtain the equation $\frac{1}{4}x^2+1 = x$. Hence $x^2-4x+4 = 0$, yielding $x = 2$. Then $\frac{1}{4}+\cos\theta = 1 \Rightarrow \cos\theta = \frac{3}{4}$. Proceed as in the first solution to obtain $\cos 2\theta = \mathbf{\frac{1}{8}}$.

Problem 6. A diagonal of a regular 2017-gon is chosen at random. Compute the probability that the chosen diagonal is longer than the median length of all of the diagonals.

Solution 6. Consider the 2014 diagonals emanating from a particular vertex. By symmetry, the 1007 distinct lengths occur in pairs. The complete list of diagonal lengths consists of 2017 copies of the 1007 distinct lengths. As 1007 is odd, $\frac{1007-1}{2} = 503$ of the 1007 lengths exceed the median length and the probability is $\mathbf{\frac{503}{1007}}$. Considering that making 2017 copies of the list simply multiplies the numerator and denominator by 2017, it does not change the answer.

Problem 7. Given that $i = \sqrt{-1}$, compute $(i+1)^3(i-2)^3+3(i+1)^2(i+3)(i-2)^2+3(i+1)(i+3)^2(i-2)+(i+1)^3$.

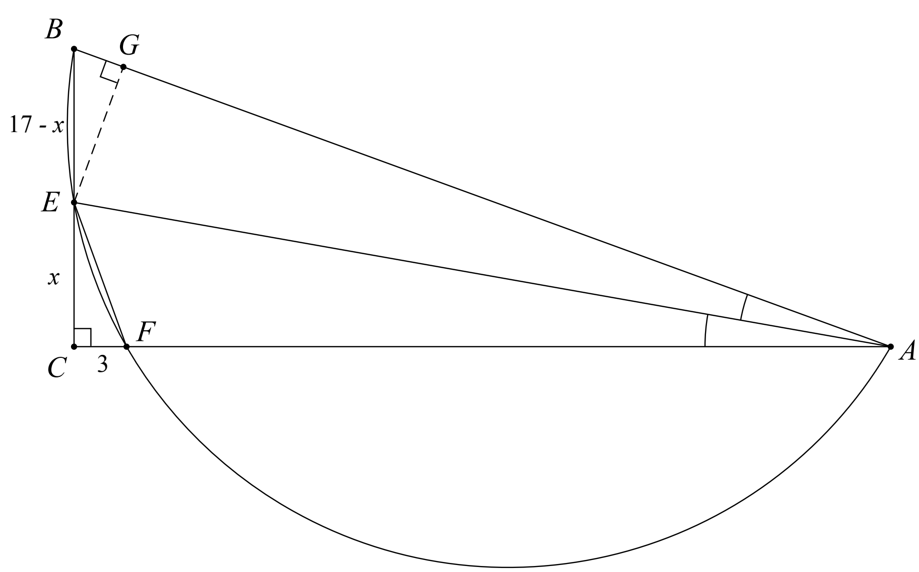
Solution 7. Let S denote the sum. Rather than expand each term, note that $[(i+1)(i-2)+(i+3)]^3 =$

$$(i+1)^3(i-2)^3+3(i+1)^2(i-2)^2(i+3)+3(i+1)(i-2)(i+3)^2+(i+3)^3$$

and $(i+1)(i-2)+(i+3) = i^2+1 = 0$. By subtraction, $S-0 = (i+1)^3-(i+3)^3 = (i^3+3i^2+3i+1)-(i^3+9i^2+27i+27) = -6i^2-24i-26 = \mathbf{-24i-20}$.

Problem 8. In triangle ABC, $\text{m}\angle C = 90°$ and $BC = 17$. Point E lies on side $\overline{BC}$ such that $\text{m}\angle CAE = \text{m}\angle EAB$. The circumcircle of triangle ABE passes through a point F on side $\overline{AC}$. Given that $CF = 3$, compute AB.

Solution 8. First note that because $\text{m}\angle CAE = \text{m}\angle EAB$, minor arcs $\widehat{FE}$ and $\widehat{EB}$ have the same length, and so $EF = EB$. Then if $CE = x$, $EB = 17-x = EF$, so by the Pythagorean Theorem, $x^2+3^2 = (17-x)^2$, or $34x = 280$, yielding $x = \frac{140}{17}$ and $EB = 17-x = \frac{149}{17}$. Proceed by constructing G on $\overline{AB}$ such that $\overline{EG} \perp \overline{AB}$, yielding the diagram below.



Then $\triangle ACE \cong \triangle AGE$ because both are right triangles sharing hypotenuse $\overline{AE}$ and with congruent angles at A. So $EG = CE$. Hence $\triangle CEF \cong \triangle GEB$ and $GB = CF = 3$. However, $AG = AC$ implies that $GB = AB - AC$. Then $AB^2 - AC^2 = BC^2 = 289$ and $AB^2 - AC^2 = (AB + AC)(AB - AC)$ yields $AB + AC = \frac{289}{3}$. Then $AB = \frac{(AB-AC)+(AB+AC)}{2} = \frac{3+289/3}{2} = \boldsymbol{\frac{149}{3}}$.

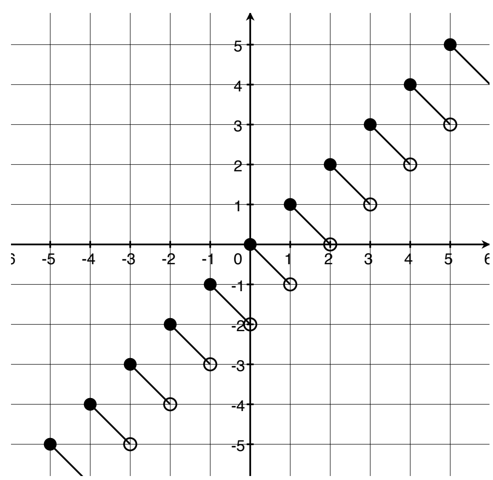
Alternate Solution: As in the first solution, compute $CE = \frac{140}{17}$ and $EB = \frac{149}{17}$. Then by the Power of the Point theorem, $CE \cdot CB = CF \cdot CA$, so $\frac{140}{17} \cdot 17 = 3 \cdot CA$, yielding $CA = \frac{140}{3}$. From this point there are at least two ways to compute AB. Using the Pythagorean Theorem is messy because it results in $AB^2 = \frac{22201}{9}$, although a reasonable guess at $\sqrt{22201}$ is 149 (because 22201 is just slightly less than $22500 = 150^2$ and the last digit of 22201 is 1), which turns out to be correct. Using the Angle Bisector Theorem yields $\frac{AB}{AC} = \frac{BE}{EC} = \frac{149}{140}$, and given that $AC = \frac{140}{3}$, $AB = \boldsymbol{\frac{149}{3}}$.

Problem 9. Let S be the set of divisors of $67 \cdot 9! + 27 \cdot 8!$. Compute the median of S.

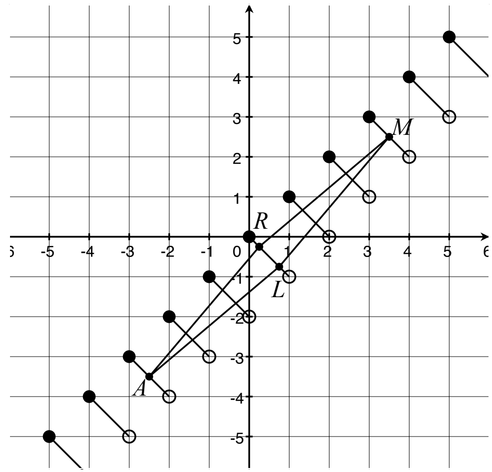
Solution 9. Rewrite $67 \cdot 9! + 27 \cdot 8!$ as $(67 \cdot 9 + 27)8! = 630 \cdot 8! = 5040 \cdot 7! = (7!)^2$. Divisors occur in pairs except for the central divisor, $7! = \mathbf{5040}$, which is the median of S.

Problem 10. Rhombus $ARML$ has its vertices on the graph of $y = \lfloor x \rfloor - \{x\}$. Given that $[ARML] = 8$, compute the least upper bound for $\tan A$.

Solution 10. As $x = \lfloor x \rfloor + \{x\}$, $y = 2\lfloor x \rfloor - x$ or $y = 2n - x$ on the interval $n \leq x < n + 1$ for integers n. These line segments have length $\sqrt{2}$ and are separated by that same length as shown in the diagram below.



The following argument shows that for *any* rhombus to lie on this graph, one pair of opposite vertices must lie on the same diagonal segment. In fact, if U and V are points on different segments of the graph with coordinates (x_U, y_U) and (x_V, y_V) respectively, then the slope of segment $\overline{UV}$ must be positive. Suppose, without loss of generality, that $x_U < x_V$. Then $y_U < y_V$ because each segment is above the segment to its immediate left. Thus the slope of $\overline{UV}$ is $\frac{y_V - y_U}{x_V - x_U} > 0$. Because the diagonals of a rhombus are perpendicular and perpendicular segments have opposite reciprocal slopes, it is impossible that both diagonals of the rhombus have positive slopes. Hence one pair of opposite vertices of the rhombus must lie on the same diagonal segment. This argument yields a further conclusion: because the slope of the diagonal with negative slope is -1, the other two vertices must lie on a line parallel to the line $y = x$. A rhombus satisfying these conditions is shown in the diagram below.



In order for $\tan A$ to be positive, $\angle A$ must be acute. Hence A must be the lower-left (or upper-right) vertex of the rhombus, with R and L on the same diagonal segment; the slope of $\overline{AM}$ is 1. Because the distance between adjacent segments is $\sqrt{2}$, $AM = k\sqrt{2}$ for some integer k, and because R and L lie on the same diagonal segment, $RL < \sqrt{2}$. (The inequality is strict because the right endpoint of each segment is *not* included in the graph.) Because $[ARML] = \frac{1}{2} AM \cdot RL = 8$, $RL = \frac{16}{k\sqrt{2}} = \frac{8\sqrt{2}}{k}$. Hence $k > 8$, and to maximize $\angle A$, it suffices to maximize RL. If $k = 9$, however, the midpoint of $\overline{AM}$ does not lie on any segment. Thus $k = 10$ and $RL = \frac{8\sqrt{2}}{10} = \frac{4\sqrt{2}}{5}$ while $AM = 10\sqrt{2}$. Hence $\tan \frac{A}{2} = \frac{RL/2}{AM/2} = \frac{RL}{AM} = \frac{2}{25}$. Using the double-angle identity,

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} = \frac{\frac{4}{25}}{1 - \frac{4}{625}} = \frac{100}{625 - 4} = \mathbf{\frac{100}{621}}.$$

Note that this least upper bound for $\tan A$ is in fact attainable, although the problem was stated to avoid having to assume that conclusion.

9 Relay Problems

Relay 1-1. Compute the greatest integer b for which $\log_b(136!) - \log_b(135!) - \log_b(17)$ is an integer.

Relay 1-2. Let $T = TNYWR$. Given that x and y are integers satisfying $x^2 - y^2 = T$, compute the least possible value of $x^2 + y^2$.

Relay 1-3. Let $T = TNYWR$. Rectangle $SHAW$ has side lengths $SH = 24$ and $SW = T$. Point D lies on $\overline{AW}$ such that $\overline{HD} \perp \overline{SA}$. Point E is the intersection of $\overline{HD}$ and $\overline{SA}$. Compute DE.

Relay 2-1. Compute the number of three-digit positive integers that are divisible by 11 and have middle digit 6.

Relay 2-2. Let $T = TNYWR$. Compute the least positive integer N such that when a fair N-sided die whose faces are numbered consecutively from 1 through N is rolled once, the probability of rolling a factor of N is less than $\dfrac{1}{T}$.

Relay 2-3. Let $T = TNYWR$. Given that $1 - r + r^3 - r^4 + r^6 - r^7 + \cdots = \dfrac{T^2}{1 + T + T^2}$, compute r.

10 Relay Answers

Answer 1-1. 8

Answer 1-2. 10

Answer 1-3. $\frac{125}{78}$ $\left(\text{or } 1\frac{47}{78}\right)$

Answer 2-1. 8

Answer 2-2. 17

Answer 2-3. $\frac{1}{17}$

11 Relay Solutions

Relay 1-1. Compute the greatest integer b for which $\log_b(136!) - \log_b(135!) - \log_b(17)$ is an integer.

Solution 1-1. Using the property $\log_b x - \log_b y = \log_b \frac{x}{y}$, $\log_b(136!) - \log_b(135!) = \log_b\left(\frac{136!}{135!}\right) = \log_b(136)$ and $\log_b(136) - \log_b(17) = \log_b(136/17) = \log_b(8)$. Therefore the greatest integer b for which $\log_b(8)$ is an integer is $b = \mathbf{8}$.

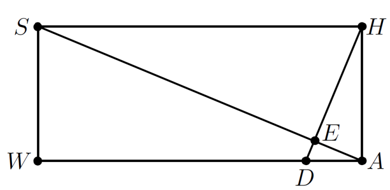
Relay 1-2. Let $T = TNYWR$. Given that x and y are integers satisfying $x^2 - y^2 = T$, compute the least possible value of $x^2 + y^2$.

Solution 1-2. Note that $T = (x+y)(x-y)$. Let $m = x+y$ and $n = x-y$. Solving the system for x and y yields $(x,y) = \left(\frac{m+n}{2}, \frac{m-n}{2}\right)$. Therefore x and y are both integers if and only if m and n have the same parity. Thus it suffices to consider all factorizations of $T = mn$ such that m and n have the same parity and $m \geq n \geq 1$. (The reader can verify that the value of $x^2 + y^2$ does not change in the cases where $n = x+y$ and $m = x-y$ or where $-m = x+y$ and $-n = x-y$.) With $T = 8$, the only ordered pair of positive integers (m,n) of the same parity with $m \geq n$ and $mn = 8$ is $(4,2)$, resulting in $(x,y) = (3,1)$. Hence $x^2 + y^2 = 3^2 + 1^2 = \mathbf{10}$.

Alternate Solution: Noting that $x^2 + y^2 = x^2 - y^2 + 2y^2 = T + 2y^2$, if T is fixed, then $x^2 + y^2$ is minimized when y^2 is minimized. With $T = 8$, the integers $x = 3$ and $y = 1$ satisfy the given equation, so the least possible value of $x^2 + y^2$ is $\mathbf{10}$.

Relay 1-3. Let $T = TNYWR$. Rectangle $SHAW$ has side lengths $SH = 24$ and $SW = T$. Point D lies on $\overline{AW}$ such that $\overline{HD} \perp \overline{SA}$. Point E is the intersection of $\overline{HD}$ and $\overline{SA}$. Compute DE.

Solution 1-3. Note that $\triangle SHE \sim \triangle ADE$ because $\angle HSE \cong \angle DAE$ and $\angle HES \cong \angle DEA$. Let $HE = a$ and $SE = b$. Then for some positive real number $k < 1$, it follows that $DE = ka$, $AE = kb$, and $AD = k \cdot SH = 24k$. Also note that $\triangle SHE \sim \triangle HAE \sim \triangle SAH$, hence $HE/SE = AE/HE = AH/SH$, which implies $a/b = kb/a = T/24$. Using $a/b = kb/a$, conclude that $k = a^2/b^2 = (T/24)^2$. Using $a/b = T/24$, conclude that $b = 24a/T$, and because $a^2 + b^2 = 24^2$ (by the Pythagorean Theorem), it follows that $a = \frac{24T}{\sqrt{T^2+24^2}}$. Hence $DE = ka = \frac{T^3}{24\sqrt{T^2+24^2}}$ $(\star)$. With $T = 10$, the denominator of the fraction in $(\star)$ simplifies to $24 \cdot 26$ and the numerator is $10^3 = 1000$. Because 1000 and 24 have a common factor of 8, the expression for DE simplifies to $\frac{125}{3 \cdot 26} = \mathbf{\frac{125}{78}}$.



Relay 2-1. Compute the number of three-digit positive integers that are divisible by 11 and have middle digit 6.

Solution 2-1. Represent the number as $\underline{A}\ \underline{6}\ \underline{B}$. Then $A - 6 + B$ must be a multiple of 11. This can only happen if $A + B = 6$ or $A + B = 17$. The first case has six possibilities ($A = 1, 2, 3, 4, 5, 6$) and the latter case has two possibilities ($A = 8, 9$) for a total of $\mathbf{8}$.

Relay 2-2. Let $T = TNYWR$. Compute the least positive integer N such that when a fair N-sided die whose faces are numbered consecutively from 1 through N is rolled once, the probability of rolling a factor of N is less than $\dfrac{1}{T}$.

Solution 2-2. Let $P(N)$ denote the probability of rolling a factor of N on a fair N-sided die. Then $P(1) = 1$, $P(2) = 1$, $P(3) = \frac{2}{3}$, $P(4) = \frac{3}{4}$, $P(5) = \frac{2}{5}$, and $P(6) = \frac{2}{3}$. More generally, $P(N) = \frac{2}{N}$ when N is prime and $P(N)$ is greater than $\frac{2}{N}$ when N is composite. Try the least prime N such that $\frac{2}{N} < \frac{1}{T}$, or $N > 2T$. As $T = 8$, try $N = 17$. It is straightforward to see that $P(17) = \frac{2}{17} < \frac{1}{8}$, and it can also easily be checked that for $1 \leq N \leq 16$, $P(N) \geq \frac{1}{T}$. Thus $N = \mathbf{17}$.

Relay 2-3. Let $T = TNYWR$. Given that $1 - r + r^3 - r^4 + r^6 - r^7 + \cdots = \dfrac{T^2}{1 + T + T^2}$, compute r.

Solution 2-3. The left-hand side of the given equation is $(1 - r) + r^3(1 - r) + r^6(1 - r) + \cdots = \frac{1-r}{1-r^3} = \frac{1}{1+r+r^2}$ for $|r| < 1$. The right-hand side of the given equation is $\frac{1}{\frac{1}{T^2} + \frac{1}{T} + 1}$, so $r = \frac{1}{T}$ is a solution as long as $|T| > 1$. More generally, $\frac{1}{1+r+r^2} = \frac{T^2}{1+T+T^2} \Rightarrow r^2 + r + 1 = \frac{1}{T^2} + \frac{1}{T} + 1 \Rightarrow r^2 + r - \frac{1}{T^2} - \frac{1}{T} = 0 \Rightarrow (r - \frac{1}{T})(r + 1 + \frac{1}{T}) = 0$ so r is either $\frac{1}{T}$ or $-1 - \frac{1}{T}$. As $T = 17$, the series only converges for $r = \dfrac{\mathbf{1}}{\mathbf{17}}$.

12 Super Relay

1. Compute the units digit of $17^1 + 17^2 + 17^3 + \ldots + 17^{16} + 17^{17}$.

2. Let $T = TNYWR$. Given that the polynomial $x^2 - 17x + TK$ can be factored over the integers and that K is an integer, compute the greatest possible value of K.

3. Let $T = TNYWR$ and let $K = 17 - T$. Let

$$P = \sqrt{K + \sqrt{K + \sqrt{K + \ldots}}} \qquad \text{and} \qquad J = 1 + \cfrac{K}{1 + \cfrac{K}{1 + \cfrac{K}{1 + \ldots}}}.$$

Compute $\lfloor P - J \rfloor$.

4. Let $T = TNYWR$. In $\triangle PJK$, each of $m\angle P$ and $m\angle J$ is an integral multiple of 17° and $m\angle P \geq m\angle J$. Let S be the number of triangles that satisfy these conditions, where no two of these triangles are similar to one another. Compute $S + T$.

5. Let $T = TNYWR$. Square $IJKL$ has area T. Diagonal KI is extended past I to point P such that $PK = 17\sqrt{2}$. Points E and O lie in the plane such that $EIOP$ is a square and E and L lie on the same side of $\overline{PK}$. Compute the perimeter of trapezoid $POLE$.

6. Let $T = TNYWR$. Donald and John each play the trumpet. They take turns (starting with Donald), where each person plays a note subject to the constraint that neither person can play the same note that was last played by the other person. There are a total of 17 different notes they can play and between them, they play a total of $\lfloor T + 1 \rfloor$ notes. Given that the number of melodies they can play can be expressed in the form $w \cdot x^y$, where w, x, and y are positive integers and w and x are relatively prime, compute the least possible value of $w + x + y$.

7. Let $T = TNYWR$. A triangle is similar to an 8-15-17 triangle and one of its sides is T. Given that the perimeter of this triangle is an integer, compute the least possible perimeter this triangle can have.

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15. Compute the number of ordered triples of integers (a, b, c) that are solutions to the equation $abc = 17$.

14. Let $T = TNYWR$ and let $P = \lfloor \sqrt{T} \rfloor$. The complex number $\dfrac{20}{i + P} + \dfrac{17}{i - P - 1}$ can be expressed in the form $J + Ki$, where J and K are real. Compute $J + K$.

13. Let $T = TNYWR$. The circle defined by $x^2 + y^2 = 17$ intersects line $\ell : y = -Tx + 3$ in two points (x_1, y_1) and (x_2, y_2). Let P be the product $\dfrac{1}{x_1} \cdot \dfrac{1}{x_2}$, let J be the sum $x_1 + x_2$, and let K be the slope of a line perpendicular to ℓ. Compute PJK.

12. Let $T = TNYWR$. The ages of Catherine, Charlie, and Elizabeth are integers. Charlie's age is $\lfloor 4T + 68 \rfloor$, and the sum of Catherine's and Elizabeth's ages is 100. Nine years ago, Elizabeth's age was a positive multiple of 17 and that number had a common factor (greater than 1) with Charlie's current age. Compute Elizabeth's current age.

11. Let $T = TNYWR$, let $K = T - 10$, and let $P = 2^{17}$. Compute $\left\lceil \log_P \left(2^0 + 2^1 + 2^2 + \ldots + 2^{K-1} + 2^K\right) \right\rceil$.

10. Let $T = TNYWR$. Iris uses voice recognition software to queue up her favorite songs. When she says a song title aloud, the probability that the software identifies the <u>wrong</u> song is $\frac{1}{100} \max\left(T, \frac{17}{T}\right)$. Given that Iris says 40 song titles from her favorite band, Iron Maiden, into her voice recognition software, compute the expected number of songs that the software <u>correctly</u> identifies.

9. Let $T = TNYWR$. Bryan has 1700 sheets of graph paper and is stuffing ARML team envelopes with 30 sheets in each envelope. Being exacting, it takes him 11 minutes and 20 seconds to count and stuff $\lfloor T - 3 \rfloor$ sheets of paper into one or more envelopes. He completely stuffs as many envelopes as he can, until he has fewer than 30 sheets remaining, at which point, he stops. Rounded to the nearest minute, compute the number of minutes it will take Bryan to complete this arduous task.

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8. Let J be the number you will receive from position 7 and let K be the number you will receive from position 9. Let A be the largest prime factor of J and let P be the largest prime factor of K. Consider the following system of equations:

$$x^3 + y = A$$
$$x^2 + 4y = P.$$

This system has three solutions, two of which are (x_1, y_1) and (x_2, y_2), where x_1 and x_2 are *non-real*. Compute the value of $x_1 + x_2$.

13 Super Relay Answers

1. 7

2. 10

3. 0

4. 25

5. 54

6. 87

7. 232

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15. 12

14. -1

13. $-\frac{3}{4}$

12. 94

11. 5

10. 38

9. 544

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8. $-\frac{11}{4}$

14 Super Relay Solutions

Problem 1. Compute the units digit of $17^1 + 17^2 + 17^3 + \ldots + 17^{16} + 17^{17}$.

Solution 1. The units digit of 17^n is the same as the units digit of 7^n. The sequence of units digits of 7^n is readily found to be $7, 9, 3, 1, 7, 9, \ldots$, which is periodic with period 4. Moreover, the sum of any four consecutive terms of this sequence has the same units digit as $7 + 9 + 3 + 1 = 20$. Thus $17^1 + 17^2 + 17^3 + \ldots + 17^{16}$ has a units digit of 0, and the units digit of the given sum is the same as the units digit of 17^{17}, which is **7**.

Problem 2. Let $T = TNYWR$. Given that the polynomial $x^2 - 17x + TK$ can be factored over the integers and that K is an integer, compute the greatest possible value of K.

Solution 2. Suppose that $x^2 - 17x + TK$ can be factored as $(x - r)(x - s)$. By Vieta's Formulas, $r + s = 17$ and $rs = TK$. Now consider different possibilities for the sign of T. If $T = 0$, then K is not uniquely determined. If $T < 0$, then $K = \frac{rs}{T} = \frac{r(17-r)}{T}$ can be written as a quadratic function of r with a positive leading coefficient, so K has no maximum value. If $T > 0$, then to determine the possible values of K, examine pairs of positive integers r and s that sum to 17 such that T divides rs. With $T = 7$, consider $r = 7, s = 10$, so that $K = \frac{rs}{T} = 10$, or $r = 14, s = 3$, so that $K = \frac{rs}{T} = 6$. Thus the greatest possible value of K is **10**.

Problem 3. Let $T = TNYWR$ and let $K = 17 - T$. Let

$$P = \sqrt{K + \sqrt{K + \sqrt{K + \ldots}}} \qquad \text{and} \qquad J = 1 + \cfrac{K}{1 + \cfrac{K}{1 + \cfrac{K}{1 + \ldots}}}.$$

Compute $\lfloor P - J \rfloor$.

Solution 3. To compute P, write $P = \sqrt{K + P}$. This is equivalent to $P^2 - P = K$. To compute J, write $J = 1 + \frac{K}{J}$. This is equivalent to $J^2 - J = K$. Thus P and J are both roots of the equation $x^2 - x - K = 0$. The roots of this equation are $x = \frac{1 \pm \sqrt{1+4K}}{2}$. If $K = 0$, then by inspection, $P = 0$ and $J = 1$. If $K > 1$, then $P = J = \frac{1+\sqrt{1+4K}}{2}$ because the other root is negative, yet $P > 0, J > 0$. With $T = 10$, $K = 17 - 10 = 7 > 0$. Thus $P - J = 0$, and the answer is **0**.

Problem 4. Let $T = TNYWR$. In $\triangle PJK$, each of $\mathrm{m}\angle P$ and $\mathrm{m}\angle J$ is an integral multiple of 17° and $\mathrm{m}\angle P \geq \mathrm{m}\angle J$. Let S be the number of triangles that satisfy these conditions, where no two of these triangles are similar to one another. Compute $S + T$.

Solution 4. Let $p = \mathrm{m}\angle P = (17a)^\circ$ and $j = \mathrm{m}\angle J = (17b)^\circ$, where a and b are positive integers. Then $a \geq b$ and $2 \leq n \leq 10$, where $n = a + b$. By examining small values of n, conclude that the equation $a + b = n$ has $\lfloor \frac{n}{2} \rfloor$ solutions with $a \geq b$. Thus there are $2(1 + 2 + 3 + 4) + 5 = 25$ possible solutions, so $S = 25$. With $T = 0$, $S + T = \mathbf{25}$.

Problem 5. Let $T = TNYWR$. Square $IJKL$ has area T. Diagonal KI is extended past I to point P such that $PK = 17\sqrt{2}$. Points E and O lie in the plane such that $EIOP$ is a square and E and L lie on the same side of $\overline{PK}$. Compute the perimeter of trapezoid $POLE$.

Solution 5. The side of square $IJKL$ is $\sqrt{T}$ and the diagonal's length is $\sqrt{2T}$. Thus the diagonal of square $EIOP$ has length $17\sqrt{2}-\sqrt{2T}$, hence the side of square $EIOP$ is $17-\sqrt{T}$. Note that $OL = OI + IL = (17-\sqrt{T})+\sqrt{T} = 17$, $PO = EP = EI = 17-\sqrt{T}$, and $LE = \sqrt{EI^2 + IL^2}$. Thus the perimeter of trapezoid $POLE$ is $51 - 2\sqrt{T} + \sqrt{2T - 34\sqrt{T} + 289}$. With $T = 25$, $\sqrt{T} = 5$. Thus square $EIOP$ has side length $17-5 = 12$, $\triangle LIE$ is a 5-12-13 triangle, and the perimeter of $POLE$ is **54**.

Problem 6. Let $T = TNYWR$. Donald and John each play the trumpet. They take turns (starting with Donald), where each person plays a note subject to the constraint that neither person can play the same note that was last played by the other person. There are a total of 17 different notes they can play and between them, they play a total of $\lfloor T+1 \rfloor$ notes. Given that the number of melodies they can play can be expressed in the form $w \cdot x^y$, where w, x, and y are positive integers and w and x are relatively prime, compute the least possible value of $w + x + y$.

Solution 6. The first note can be played in 17 ways. Each note after the first can be played in 16 ways. Thus the total number of melodies that Donald and John can play is $17 \cdot (2^4)^{\lfloor T \rfloor}$. Thus $w = 17$. With $T = 54$, the possible values of x are numbers of the form 2^d where d is a divisor of $216 = 4 \cdot 54$. Some quick computations reveal that $x + y = 2^d + \frac{216}{d}$ is minimized when $d = 4$ so that $x = 16$ and $y = 54$. Thus the least possible value of $w + x + y$ is $17 + 16 + 54 = \mathbf{87}$.

Problem 7. Let $T = TNYWR$. A triangle is similar to an 8-15-17 triangle and one of its sides is T. Given that the perimeter of this triangle is an integer, compute the least possible perimeter this triangle can have.

Solution 7. Let the triangle have sides $8x, 15x, 17x$, where $x > 0$. Then the triangle's perimeter is $P = 40x$ and $T \in \{8x, 15x, 17x\}$. If $T = 8x$, then $P = 5(8x) = 5T$. If $T = 15x$, then $P = \frac{8T}{3}$. Finally, if $T = 17x$, then $P = \frac{40T}{17}$. Note that $\frac{40}{17} < \frac{8}{3} < 5$. With $T = 87$, note that 17 does not divide 87, hence P is not an integer. But because 3 divides 87, $P = \frac{8T}{3}$ is integer and is equal to **232**.

Problem 15. Compute the number of ordered triples of integers (a, b, c) that are solutions to the equation $abc = 17$.

Solution 15. If a, b, and c are all positive, then two of the variables must be 1 and the third variable must be 17. This gives 3 solutions. If not all of a, b, and c are positive, then exactly two variables must be negative. For each of the 3 positive solutions, there are 3 choices for which variables could be negative. Thus the answer is $3 + 3 \cdot 3 = \mathbf{12}$.

Problem 14. Let $T = TNYWR$ and let $P = \lfloor \sqrt{T} \rfloor$. The complex number $\dfrac{20}{i+P} + \dfrac{17}{i-P-1}$ can be expressed in the form $J + Ki$, where J and K are real. Compute $J + K$.

Solution 14. Note that $\dfrac{20}{i+P} = \dfrac{20}{i+P} \cdot \dfrac{-i+P}{-i+P} = \dfrac{-20i+20P}{P^2+1}$. Similarly, $\dfrac{17}{i-P-1} = \dfrac{17}{i-P-1} \cdot \dfrac{-i-P-1}{-i-P-1} = \dfrac{-17i-17(P+1)}{(P+1)^2+1}$. With $T = 12$, $P = \lfloor \sqrt{12} \rfloor = 3$. Thus the given expression is equal to $\dfrac{-20i+60}{10} + \dfrac{-17i-68}{17} = (-2i+6) + (-i-4) = 2 - 3i$. Hence $J = 2, K = -3$, and $J + K = \mathbf{-1}$.

Problem 13. Let $T = TNYWR$. The circle defined by $x^2 + y^2 = 17$ intersects line $\ell : y = -Tx + 3$ in two points (x_1, y_1) and (x_2, y_2). Let P be the product $\dfrac{1}{x_1} \cdot \dfrac{1}{x_2}$, let J be the sum $x_1 + x_2$, and let K be the slope of a line perpendicular to ℓ. Compute PJK.

Solution 13. Because the slope of ℓ is $-T$, it follows that $K = \frac{-1}{-T} = \frac{1}{T}$. Substitute $y = -Tx+3$ into the equation $x^2 + y^2 = 17$ to obtain $x^2 + (-Tx+3)^2 = 17$ or $(1+T^2)x^2 - 6Tx - 8 = 0$. The two roots of this equation are x_1 and x_2. By Vieta's Formulas, $x_1 + x_2 = \frac{6T}{1+T^2}$ and $x_1x_2 = \frac{-8}{1+T^2}$. Hence $P = \frac{1+T^2}{-8}$ and $PJ = \frac{6T}{-8}$. Finally, $PJK = \left(\frac{6T}{-8}\right) \cdot \frac{1}{T} = -\mathbf{\frac{3}{4}}$ (independent of T).

Problem 12. Let $T = TNYWR$. The ages of Catherine, Charlie, and Elizabeth are integers. Charlie's age is $\lfloor 4T + 68 \rfloor$, and the sum of Catherine's and Elizabeth's ages is 100. Nine years ago, Elizabeth's age was a positive multiple of 17 and that number had a common factor (greater than 1) with Charlie's current age. Compute Elizabeth's current age.

Solution 12. List the multiples of 17 that are less than 100: $17, 34, 51, 68, 85$. Thus Elizabeth's current age is one of $26, 43, 60, 77$, or 94. With $T = -\frac{3}{4}$, Charlie's current age is $65 = 5 \cdot 13$. The only possible age of Elizabeth nine years ago that shares a common factor greater than 1 with 65 is 85. Hence Elizabeth's current age is **94**.

Problem 11. Let $T = TNYWR$, let $K = T-10$, and let $P = 2^{17}$. Compute $\left\lceil \log_P \left(2^0 + 2^1 + 2^2 + \ldots + 2^{K-1} + 2^K\right) \right\rceil$.

Solution 11. By the formula for the sum of the terms of a geometric series, it follows that $2^0 + 2^1 + 2^2 + \ldots + 2^{K-1} + 2^K = 2^{K+1} - 1$. Thus $\log_P \left(2^0 + 2^1 + 2^2 + \ldots + 2^{K-1} + 2^K\right) \lessapprox \log_{2^{17}} 2^{K+1} = \frac{K+1}{17}$. With $T = 94$, $K = 84$, and $\frac{K+1}{17} = 5$. Thus $\left\lceil \log_P \left(2^0 + 2^1 + 2^2 + \ldots + 2^{K-1} + 2^K\right) \right\rceil = \mathbf{5}$.

Problem 10. Let $T = TNYWR$. Iris uses voice recognition software to queue up her favorite songs. When she says a song title aloud, the probability that the software identifies the <u>wrong</u> song is $\frac{1}{100} \max\left(T, \frac{17}{T}\right)$. Given that Iris says 40 song titles from her favorite band, Iron Maiden, into her voice recognition software, compute the expected number of songs that the software <u>correctly</u> identifies.

Solution 10. When a song title is read aloud, the probability that the software identifies the correct song is $1 - \frac{1}{100} \max\left(T, \frac{17}{T}\right)$. Thus the expected number of correctly identified songs is $40 - \max\left(\frac{2T}{5}, \frac{34}{5T}\right)$. With $T = 5$, $\max\left(\frac{2T}{5}, \frac{34}{5T}\right) = \max\left(2, \frac{34}{25}\right) = 2$. Thus the answer is $40 - 2 = \mathbf{38}$.

Problem 9. Let $T = TNYWR$. Bryan has 1700 sheets of graph paper and is stuffing ARML team envelopes with 30 sheets in each envelope. Being exacting, it takes him 11 minutes and 20 seconds to count and stuff $\lfloor T - 3 \rfloor$ sheets of paper into one or more envelopes. He completely stuffs as many envelopes as he can, until he has fewer than 30 sheets remaining, at which point, he stops. Rounded to the nearest minute, compute the number of minutes it will take Bryan to complete this arduous task.

Solution 9. Bryan can stuff $\left\lfloor \frac{1700}{30} \right\rfloor = 56$ envelopes, so he stuffs a total of $56 \cdot 30 = 1680$ sheets of paper. Note also that 11 minutes and 20 seconds is equivalent to $\frac{34}{3}$ minutes. Let S be the number of minutes Bryan needs to stuff the 1680 sheets. Then $\lfloor T-3 \rfloor / \left(\frac{34}{3}\right) = 1680/S$. Thus $S = \frac{56 \cdot 30 \cdot 34}{3 \lfloor T-3 \rfloor}$. With $T = 38$, $S = \frac{56 \cdot 30 \cdot 34}{3 \cdot 35} = 8 \cdot 2 \cdot 34 = \mathbf{544}$.

Problem 8. Let J be the number you will receive from position 7 and let K be the number you will receive from position 9. Let A be the largest prime factor of J and let P be the largest prime factor of K. Consider the following system of equations:

$$x^3 + y = A$$
$$x^2 + 4y = P.$$

This system has three solutions, two of which are (x_1, y_1) and (x_2, y_2), where x_1 and x_2 are *non-real*. Compute the value of $x_1 + x_2$.

Solution 8. Eliminate y by multiplying the first equation by 4 and subtracting the second equation to obtain $4x^3 - x^2 - 4A + P = 0$. The sum of the three roots of this equation is $-\left(\frac{-1}{4}\right) = \frac{1}{4}$. Let $f(x) = 4x^3 - x^2 - 4A + P$. As long as A and P are real, the equation $f(x) = 0$ will have at least one real root (to see this, consider the graph of $y = f(x)$). The problem implies that two of the roots of the equation $f(x) = 0$ are non-real and are equal to x_1 and x_2. To determine $x_1 + x_2$, it suffices to determine the real root r of $f(x) = 0$, and then $x_1 + x_2$ will be equal to $\frac{1}{4} - r$. With $J = 232 = 2^3 \cdot 29$ and $K = 544 = 2^5 \cdot 17$, it follows that $A = 29$, $P = 17$, and $f(x) = 4x^3 - x^2 - 99$. Note that $f(3) = 108 - 9 - 99 = 0$, hence $x = 3$ is the real root of $f(x) = 0$, and it can be verified that the quadratic factor of $f(x)$ (i.e., $4x^2 + 11x + 33$) has a negative discriminant, hence no real roots. Thus the answer is $\frac{1}{4} - 3 = -\mathbf{\frac{11}{4}}$.

15 Tiebreaker Problems

Problem 1. Compute the least positive N such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

Problem 2. Cube $ARMLKHJC$, with opposite faces $ARML$ and $HJCK$, is inscribed in a cone, such that A is the vertex of the cone, edges $\overline{AR}, \overline{AL}, \overline{AH}$ lie on the surface of the cone, and vertex C, diagonally opposite A, is on the base of the cone. Given that $AR = 6$, compute the radius of the cone.

Problem 3. Given that $\log_{b^3} a^2 + \log_{b^9} a^4 + \log_{b^{27}} a^8 + \cdots = 1$, compute $\log_b a$.

16 Tiebreaker Answers

Answer 1. 137

Answer 2. $6\sqrt{6}$

Answer 3. $\frac{1}{2}$ (or 0.5)

17 Tiebreaker Solutions

Problem 1. Compute the least positive N such that there exists a quadratic polynomial $f(x)$ with integer coefficients satisfying

$$f(f(1)) = f(f(5)) = f(f(7)) = f(f(11)) = N.$$

Solution 1. Because $f(x)$ is a quadratic polynomial, the graph of $y = f(x)$ is symmetric about some vertical line; for each output, there are no more than two corresponding inputs. Thus if $f(p) = f(q) = f(r) = f(s)$, then the variables p, q, r, s can take on at most two distinct values. Hence the set $\{f(1), f(5), f(7), f(11)\}$ contains at most two distinct integers. However, the same logic shows that the equation $f(p) = f(q) = f(r)$ has no solutions when p, q, r are required to be distinct. Hence it is impossible that three of the values $f(1), f(5), f(7), f(11)$ be the same. Thus the set $\{f(1), f(5), f(7), f(11)\}$ contains exactly two distinct integers, dividing the expressions $f(1), f(5), f(7), f(11)$ into two pairs of equal values. Because of the symmetry of the graph, there must be a unique value of h such that $f(x) = f(2h - x)$ for all x. It follows that $f(1) = f(11)$ and $f(5) = f(7)$, with $h = 6$. (Any other pairing yields a contradiction: for example, if $f(1) = f(7)$, then $h = 4$, but the equality of the other pair $f(5) = f(11)$ yields $h = 8$.) Hence $f(x)$ is of the form $f(x) = a(x - 6)^2 + k$, and $f(f(x)) = a\big(a(x-6)^2 + k - 6\big)^2 + k$.

The following argument shows that N is minimal when $a = 1$. From the condition that $f(f(1)) = f(f(5)) = f(f(7)) = f(f(11))$, it follows that

$$a(f(1) - 6)^2 + k = a(f(5) - 6)^2 + k = a(f(7) - 6)^2 + k = a(f(11) - 6)^2 + k.$$

Simplifying yields the following equation:

$$\pm(f(1) - 6) = \pm(f(5) - 6) = \pm(f(7) - 6) = \pm(f(11) - 6).$$

Moreover, it follows from the above symmetry argument that $f(5) = f(7) \neq f(1) = f(11)$. Thus $f(1) - 6 = -(f(5) - 6)$. Because $f(1) = 25a + k$, and $f(5) = a + k$, it follows that $25a + k - 6 = -a - k + 6$, which is equivalent to $k = 6 - 13a$. (Notice that the equation $f(7) - 6 = -(f(11) - 6)$ adds no information and yields an equivalent equation.) Now, computing $N = f(f(5))$,

$$\begin{aligned} f(f(5)) &= f(a + k) \\ &= a \cdot (a + k - 6)^2 + k \\ &= a \cdot (-12a)^2 + (6 - 13a) \\ &= 144a^3 - 13a + 6. \end{aligned}$$

So the problem reduces to finding the minimum positive value of the expression $144a^3 - 13a + 6$ when a is an integer. First note that $a \neq 0$ because f is a quadratic polynomial. Next, if $a \leq -1$, then $a^3 \leq a$, so

$$N = 144a^3 - 13a + 6 \leq 144a - 13a + 6 = 131a + 6 \leq -131 + 6 < 0,$$

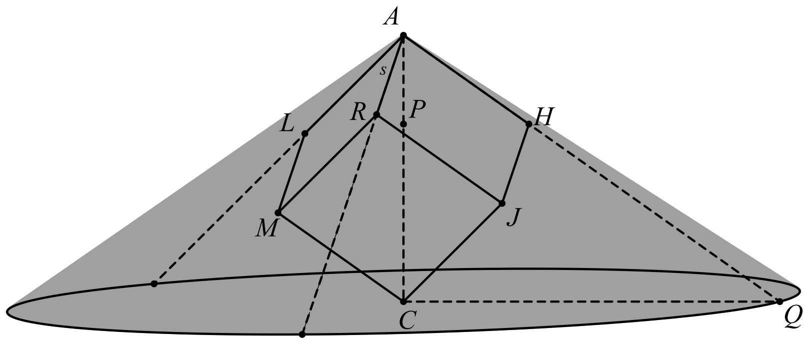
which violates the condition that N be positive. Because a is an integer, no values between -1 and 1 need be considered. For $a \geq 1$, $a^3 \geq a$, so

$$N = 144a^3 - 13a + 6 \geq 144a - 13a + 6 = 131a + 6,$$

with equality precisely when $a = 1$. Thus, with $a = 1$, $N = 137$, and for any other positive value of a, N will be strictly larger. Hence the least positive value of N is **137.**

Problem 2. Cube $ARMLKHJC$, with opposite faces $ARML$ and $HJCK$, is inscribed in a cone, such that A is the vertex of the cone, edges $\overline{AR}, \overline{AL}, \overline{AH}$ lie on the surface of the cone, and vertex C, diagonally opposite A, is on the base of the cone. Given that $AR = 6$, compute the radius of the cone.

Solution 2. The cone and cube are pictured in the diagram below.



First note that point C must be the center of the cone's base because of the rotational symmetry of vertices R, L, H about the axis of the cone. Thus AC is the height of the cone. Furthermore, plane RLH is parallel to the cone's base (again, because of the rotational symmetry of vertices R, L, H). Let P be the point where $\overline{AC}$ intersects plane RLH, and let $s = AR$. Because $\triangle RLH$ is equilateral with side length $s\sqrt{2}$, conclude that P is the centroid of $\triangle RLH$, and $PH = \frac{2}{3}\left(\frac{s\sqrt{2}}{2} \cdot \sqrt{3}\right) = \frac{s\sqrt{6}}{3}$. Use the Pythagorean Theorem on $\triangle APH$ to compute $AP = \frac{s\sqrt{3}}{3}$. Extend $\overrightarrow{AH}$ to meet the bottom of the cone at Q. Because $AC = s\sqrt{3}$ and $\triangle APH \sim \triangle ACQ$, it follows that $\frac{CQ}{PH} = \frac{AC}{AP} = 3$, hence $CQ = s\sqrt{6}$. Substitute $s = 6$ to obtain $CQ = \mathbf{6\sqrt{6}}$.

Problem 3. Given that $\log_{b^3} a^2 + \log_{b^9} a^4 + \log_{b^{27}} a^8 + \cdots = 1$, compute $\log_b a$.

Solution 3. Apply logarithm laws to the left-hand side of the given equation:

$$\begin{aligned}\log_{b^3} a^2 + \log_{b^9} a^4 + \log_{b^{27}} a^8 + \cdots &= \frac{2}{3}\log_b a + \frac{4}{9}\log_b a + \frac{8}{27}\log_b a + \cdots \\ &= (\log_b a)\left(\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots\right),\end{aligned}$$

which is an infinite geometric series with first term $\frac{2}{3}\log_b a$ and common ratio $\frac{2}{3}$. Hence the sum of the series is $2\log_b a = 1$. Therefore $\log_b a = \mathbf{\frac{1}{2}}$.