2015 ARML Local Problems and Solutions

Team Round Solutions

T-1 For a number x, let S_x be the set $\{5, 5, 5, 7, 8, 13, x\}$. Compute the sum of the values of x such that the mean, median, and mode of S_x can be arranged to form an *increasing* arithmetic sequence.

T-1 Solution The mode of S_x is 5. The median is 5 if $x \leq 5$, x if $5 \leq x \leq 7$, and 7 if $x \geq 7$. The mean of S_x is $M = \frac{43+x}{7}$. If $x \geq 7$, then $M > 7$ so the arithmetic sequence is $5, 7, M$, which means $M = 9$, so $x = 20$. If x is between 5 and 7, then the arithmetic sequence is either $5, x, M$ or $5, M, x$. The former sequence has the solution $M = 7$ and $x = 6$, while the latter sequence is impossible for x between 5 and 7. The sum of the values of x is 26.

T-2 Compute the value of N for which $\dfrac{1}{\log_2 100} + \dfrac{1}{\log_3 100} + \dfrac{1}{\log_6 100} + \dfrac{1}{\log_9 100} = \dfrac{2}{\log_N 100}$.

T-2 Solution Changing bases, the equation is equivalent to $\dfrac{\log 2}{\log 100} + \dfrac{\log 3}{\log 100} + \dfrac{\log 6}{\log 100} + \dfrac{\log 9}{\log 100} = \dfrac{2 \log N}{\log 100}$. Multiplying through by $\log 100$, the equation becomes $\log(2 \cdot 3 \cdot 6 \cdot 9) = \log N^2$, so $N = 18$.

T-3 A non-negative integer is called *falling* if its digits are strictly decreasing from left to right. 9876543210 is the largest falling number, 0 is the smallest. If all of the falling numbers were written in a list from smallest to largest, compute the 1000^{th} number on that list.

T-3 Solution There are $2^{10} = 1024$ falling numbers in total. One has 10 digits, 10 have 9 digits, and $C(10, 2) = 45$ have 8 digits. Therefore, the answer will be the 33^{rd} largest among those with eight digits. It will omit the 13^{th} smallest pair of digits, which is 5 and 2, so the answer is 98764310.

T-4 Compute the length of the interval of values of x such that $(x-1)^2 \leq 2|x-2|$.

T-4 Solution If $x \geq 2$, then $(x-1)^2 \leq 2(x-2) \to x^2 - 2x + 1 \leq 2x - 4 \to x^2 - 4x + 5 \leq 0$, which is never true as $0 \leq (x-2)^2 < (x-2)^2 + 1 = x^2 - 4x + 5$. If $x < 2$, the linear function is $2(2-x) = 4 - 2x$. $x^2 - 2x + 1 = 4 - 2x \to x^2 = 3 \to x = \pm\sqrt{3}$, so the length of the interval is $2\sqrt{3}$.

T-5 Pentagon $OPQRS$ has vertices at $O(0,0)$, $P(8,0)$, $Q(8,4)$, $R(2,4)$, and $S(0,8)$. The line $y = mx$ divides $OPQRS$ into two regions of equal area. Compute m.

T-5 Solution Consider $OPQRS$; its area is the area of a rectangle and a triangle, so $[OPQRS] = 8(4) + \frac{1}{2} \cdot 2 \cdot 4 = 36$. The line $y = mx$ leaves a trapezoid to its right. The line $y = mx$ hits side $\overline{QR}$ at a point with y-coordinate 4 and thus x-coordinate $\frac{4}{m}$. Therefore, $\frac{1}{2} \cdot 4 \cdot (8 - \frac{4}{m} + 8) = 18$, which can be solved to obtain $m = \dfrac{4}{7}$.

T-6 Compute the number of ordered triples of integers (x, y, z) such that $xy - |z| = 20$ and $|x + y| - z = 15$.

T-6 Solution For (x, y, z) to be a solution, both x and y must have the same sign and (x, y) and $(-x, -y)$ will have identical corresponding values of z. In addition, to satisfy the first equation, $xy \geq 20$.

If $z \geq 0$, then $xy - |x+y| = 5$. If $x+y \geq 0$, then $xy - (x+y) = 5$, or $(x-1)(y-1) = 6$. Therefore, $(x, y) = (7, 2), (2, 7), (4, 3)$, or $(3, 4)$. In each case, $xy < 20$ so the first equation cannot be satisfied.

If $x + y \leq 0$, then $xy + (x + y) = 5$, or $(x + 1)(y + 1) = 6$. Therefore, $(x, y) = (-7, -2), (-2, -7), (-4, -3)$, or $(-3, -4)$. In each case, $xy < 20$ so the first equation cannot be satisfied.

If $z \leq 0$, then $xy + |x+y| = 35$. If $x+y \geq 0$, then $xy + (x+y) = 35$, or $(x+1)(y+1) = 36$. If $x + y \leq 0$, then $xy - (x + y) = 35$, or $(x - 1)(y - 1) = 36$.

There are five pairs of positive integers (x, y) such that $xy \geq 20$ and $(x+1)(y+1) = 36$. They are $(2, 11)$, $(3, 8)$, $(5, 5)$, $(8, 3)$, and $(11, 2)$. There are also the corresponding pairs of negative integers, so there are a total of 10 solutions.

T-7 Define a positive integer N to be *lucky* if there exists a positive integer a such that a^7 has exactly N positive divisors. Compute the number of positive integers less than 2015 that are lucky.

T-7 Solution If $a = p_1^{e_1} \ldots p_n^{e_n}$, where the p_i are distinct primes, then the number of positive divisors of a is $\sigma(a) = \sigma(p_1^{e_1}) \ldots \sigma(p_n^{e_n}) = (e_1 + 1) \ldots (e_n + 1)$. $\sigma(a^7) = \sigma(p_1^{7e_1}) \ldots \sigma(p_n^{7e_n}) = (7e_1 + 1) \ldots (7e_n + 1)$. All values of $\sigma(a^7)$ are one more than a multiple of seven, and all numbers of the form $7k + 1$ for a non-negative integer k are equal to $\sigma(a^7)$ for $a = p^k$ for some prime p, so all numbers of the form $7k + 1$ for $k \geq 0$ are lucky. There are 288 numbers of this form less than 2015 ($287 \times 7 + 6 = 2015$).

T-8 Compute the sum of all real values of x that satisfy $(x^2 - 16x + 40)^{x^2+3x-18} = 1$.

T-8 Solution There are three scenarios, one in which a non-zero number is raised to the power of 0, one in which 1 is raised to a power, and one in which -1 is raised to a negative power. The first case requires $x^2 + 3x - 18 = 0$, and this generates the values $x = 3$ or $x = -6$, both of which cause the base to be non-zero. The second case requires $x^2 - 16x + 40 = 1 \rightarrow x^2 - 16x + 39 = 0$, and this means that $x = 13$ or $x = 3$. In the third case, $x^2 - 16x + 40 = -1$ at $x = 8 \pm \sqrt{23}$, and the exponent is not an even integer at either of these values. The solutions are $x = 13$,$x = 3$, and $x = -6$, which sum to 10.

T-9 Given square $SQUA$ with $SQ = 6$. Equilateral triangles UAR, QUE, SQC, and SAT are constructed in the exterior of $SQUA$. Compute the area of $RECT$.

T-9 Solution Note that $RECT$ is a square and its area is RE^2. Note also that RE is the long side of a triangle whose other sides are 6 and 6 and whose included angle measures 150°. Now, by the Law of Cosines, $RE^2 = 6^2 + 6^2 - 2 \cdot 6 \cdot 6 \cdot (-\frac{\sqrt{3}}{2})$, or $RE^2 = 72 + 36\sqrt{3}$. Therefore, $[RECT] = 72 + 36\sqrt{3}$.

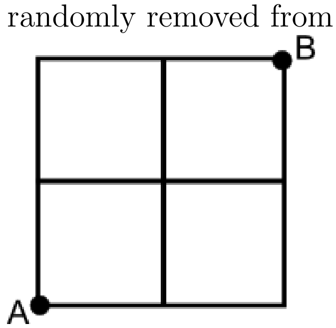
T-10 For each positive integer k, let S_k denote the increasing arithmetic sequence of 1000 integers, whose first term is 1, and whose common difference is k. For example, S_3 is the sequence $1, 4, 7, \ldots, 2998$. Compute the number of sequences in $S_1, S_2, S_3, \ldots, S_{1000}$ that contain the term 2015.

T-10 Solution Let $s_{k,n}$ denote the n^{th} element of S_k, with $s_{k,0} = 1$ for all k. $s_{k,n} = 1 + kn$, so in order for an element in S_k to be equal to 2015, $kn = 2014$ with $k \le 1000$ and $n < 1000$. $2014 = 2 \times 19 \times 53$, and $\sigma(2014) = 8$. However, two of the factors (1007 and 2014) are greater than 1000, so four pairs of factors of 2014 (k, n) are invalid, and the answer is 4.

T-11 If $g(x) = x(x-1)$, then compute the sum of all real solutions to $g(g(x)) = 30$.

T-11 Solution Let $w = g(x)$, then $g(w) = 30 \rightarrow w(w-1) = 30 \rightarrow w^2 - w - 30 = 0 \rightarrow w = -5$ or 6. Therefore, x satisfies $g(x) = -5$ or $g(x) = 6$, or $x^2 - x + 5 = 0$ or $x^2 - x - 6 = 0$. The first equation has complex solutions, and the second equation has real solutions (-2 and 3) that sum to 1.

T-12 A path along a grid's edges is called *up-right* if it consists solely of movement along segments in the upward or rightward direction. Compute the expected number of up-right paths remaining in the grid below from A to B if exactly one of the 12 segments is randomly removed from the grid.



T-12 Solution There are six up-right paths in the full grid. If one of the four segments adjacent to A or B are removed , exactly half of the up-right paths will be removed. If one of the other four boundary segments is removed, 1 up-right path is removed, if one of the four segments adjacent to the center grid point is removed, 2 up-right paths are removed. Accordingly, one-third of the time there will be 3, 4, or 5 up-right paths remaining, so the expected number of paths remaining is 4.

T-13 For a positive integer n, let $p(n)$ be the product of the digits of n and $s(n)$ be the sum of the digits of n. Compute the sum of the values of n less than 100 such that $n = p(n)+s(n)$.

T-13 Solution Let $n = 10t + u$, where t and u are the tens and ones digit of n, respectively. Then $p(n) = tu$ if $t \geq 1$ and u if $t = 0$ and $s(n) = t + u$. If $n = p(n) + s(n)$, then $10t + u = tu + t + u \rightarrow 9t = tu$, so $u = 9$ if $t \geq 1$. The solutions are all two digit numbers with 9 as a units digit, and their sum is $19 + 29 + \cdots + 99 = 531$.

T-14 A twenty-sided die is rolled twice, getting numbers x and y. Compute the probability that $\log_{10} x + \log_{10} y$ is an integer.

T-14 Solution For $\log_{10} x + \log_{10} y$ to be an integer, xy must be a power of 10. There is one pair of values (x, y) whose product is 1 $(1, 1)$, four pairs of values whose product is 10 $((1, 10), (2, 5), (5, 2), (10, 1))$, and three pairs whose product is 100 $((5, 20), (10, 10), (20, 5))$. There are 400 possible pairs, so the probability is $\frac{8}{400} = \frac{1}{50}$.

T-15 A deck of 3 red cards, 3 white cards, and 3 blue cards is shuffled and dealt into three rows of three cards. Compute the probability that there are no red cards in the first row, no white cards in the second row, and no blue cards in the third row.

T-15 Solution If k white cards appear in the first row, then $3 - k$ white cards and k red cards must appear in the third row, which means that $3 - k$ red cards and k blue cards must appear in the second row, so $3 - k$ blue cards must appear in the first row. For each row, there are $C(3, k)$ ways to arrange the cards, so the total number of arrangements of cards that satisfy this criterion is $\sum_{k=0}^{3} C(3, k)^3 = 1^3 + 3^3 + 3^3 + 1^3 = 1 + 27 + 27 + 1 = 56$. The total number of arrangements is $\frac{9!}{3!3!3!} = \frac{362880}{216} = 1680$, so the probability is $\frac{56}{1680} = \frac{1}{30}$.

Individual Round Solutions

I-1 The graph of $y = ax^2 + bx + c$ passes through $(-1, -3)$, $(1, 3)$, and $(2, 12)$. Compute abc.

I-1 Solution Substituting $x-$ and $y-$coordinates of the three points into the general equation gives us three equations: $a - b + c = -3$, $a + b + c = 3$, and $4a + 2b + c = 12$. Subtracting the first two equations yields $2b = 6 \to b = 3$. Substituting gives two equations: $a + c = 0$ and $4a + c = 6$. Subtracting these yields $3a = 6 \to a = 2 \to c = -2$. The equation is $y = 2x^2 + 3x - 2$, so the answer is -12.

I-2 Eight rational numbers are written on a blackboard. Their average is 15. One of the numbers is erased, and the average of the numbers remaining on the board is 12. Compute the value of the number that was erased.

I-2 Solution The sum of the original eight numbers is $8 \times 15 = 120$. After a number is erased, the sum of the remaining numbers is $7 \times 12 = 84$, so the erased number was 36.

I-3 Given real numbers x and y such that $x + y = 4$ and $x^3 + y^3 = 28$. Compute $x^2 + y^2$.

I-3 Solution Note that $x^3 + y^3 = 28 = (x + y)^3 - 3xy(x + y) = 64 - 3xy(4) = 64 - 12xy$, so $-36 = -12xy \to xy = 3$. Now, consider $x^2 + y^2 = (x + y)^2 - 2xy = 16 - 2 \cdot 3 = 10$. Note that this is true when $x = 1$ and $y = 3$.

I-4 Triangle NAP has side lengths $NA = 10$, $AP = 17$, and $NP = 21$. R and S are on $\overline{NP}$ such that $\overline{AR}$ is the altitude to segment $\overline{NP}$ and $\overline{AS}$ is the angle bisector from A to $\overline{NP}$. Compute RS.

I-4 Solution Either by solving a system of equations or by inspection, we see that $AR = 8$, $NR = 6$, and $RP = 15$. We also know that the angle bisector will divide $\overline{NP}$ into the same ratio as $NA : AP$. Thus, $NS : SP = 10 : 17$ and $NS + SP = 21$, so $NS = \frac{210}{27}$, and $RS = NS - NR = \frac{210}{27} - 6 = \frac{16}{9}$.

I-5 Johanna and Edward play a game with a 6-sided die. For each turn, a player rolls a die until they roll a 1, in which case they lose the game, or until they roll a 5 or 6, in which case their turn is over and the other player rolls the die under the same rules. The game continues until some player rolls a 1 and loses. If Johanna plays first, compute the probability that she wins the game.

I-5 Solution Note that on any turn of a game, a player is twice as likely to pass the die to the other player than they are to lose. Accordingly, on any given turn, a player has a 1/3 probability of losing and a 2/3 probability of passing the die to the other player. The probability that Johanna wins after one turn by each player is $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$, after two turns is $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$, and so forth. The sum of the probabilities that Johanna wins in each round is an infinite geometric series with sum $\frac{\frac{2}{9}}{1-\frac{4}{9}} = \frac{2}{5}$.

I-6 The line $x = a$ intersects the graphs of $f(x) = \log_{10}(x)$ and $g(x) = \log_{10}(x+5)$ at points B and C, respectively. If $BC = 1$, compute a.

I-6 Solution The coordinates of the two points of intersection are $(a, \log_{10}(a))$ and $(a, \log_{10}(a+5))$. Because $BC = 1$, $\log_{10}(a+5) - \log_{10}(a) = 1 \rightarrow \log_{10}\frac{a+5}{a} = 1 \rightarrow \frac{a+5}{a} = 10 \rightarrow a = \frac{5}{9}$.

I-7 Regular hexagon $HEXAGO$ has vertices $H(4,1)$ and $E(8,1)$. The hexagon lies entirely in the first quadrant. If the coordinates of X are (x_1, y_1) and the coordinates of A are (x_2, y_2), compute $x_1 + x_2 + y_1 + y_2$.

I-7 Solution Each side of the regular hexagon measures $\sqrt{(8-4)^2 + (1-1)^2} = 4$. Extend side $\overline{HE}$ and drop an altitude from X to that extension, meeting the extension at N. The measure of $\angle XEN$ is $\frac{360}{6} = 60°$, so $EN = 4\sin 30° = 2$ and $XN = 4\sin 60° = 2\sqrt{3}$, so X has coordinates $(8+2, 1+2\sqrt{3})$. Constructing a triangle in a similar way will yield $A(8, 1+2\sqrt{3}+2\sqrt{3})$. The desired sum is $10 + 8 + 1 + 2\sqrt{3} + 1 + 4\sqrt{3} = 20 + 6\sqrt{3}$.

I-8 Let $X = 20202020202020202020$, the 20-digit number consisting of 10 copies of the digits 2 and 0. Similarly, let $Y = 15151515151515151515$, the 20-digit number consisting of 10 copies of the digits 1 and 5. Compute the sum of the digits of XY.

I-8 Solution Note that $X = 20 \times Z$, where Z is the 19 digit number consisting of alternating ones and zeroes. Similarly, $Y = 15 \times Z$. Therefore, $XY = 300 \times Z^2$.
$Z^2 = 1020304050607080910090807060504030201$, so $XY = 300Z^2 = 3Z^2 \times 100$. As the 100 factor will not affect the sum of the digits,
$3Z^2 = 3060912151821242730272421181512090603$. Pairs of digits from right to left sum to $3+6+9+3+6+9+3+6+9+3+9+6+3+9+6+3+9+6+3 = 111$.

I-9 The value of x that solves $(\sqrt{2x+1} + \sqrt{x+3})^2 + \sqrt{2x+1} + \sqrt{x+3} = 12$ is of the form $A - \sqrt{B}$ where A and B are positive integers. Compute $A + B$.

I-9 Solution This problem is easier to solve if $Y = \sqrt{2x+1} + \sqrt{x+3}$, because then the given equation becomes the quadratic $Y^2 + Y - 12 = 0 \rightarrow (Y-3)(Y+4) = 0$. The second factor in Y has no real solutions for x, so we focus on the factor that implies $Y = 3$. Solving $\sqrt{2x+1} + \sqrt{x+3} = 3 \rightarrow 2x + 1 = 9 + x + 3 - 6\sqrt{x+3} \rightarrow x - 11 = -6\sqrt{x+3}$, which has two solutions: $x = 29 \pm \sqrt{828}$. Only the lesser root checks in the original equation, so $x = 29 - \sqrt{828}$ and the answer is $29 + 828 = 857$.

I-10 Square $ARML$ has unit side lengths and center O. D and E are on $\overline{AR}$, with D between A and E. If $m(\angle DOE) = 45°$ and $DE = \frac{3}{7}$, compute the largest value of ER.

I-10 Solution Let S be the midpoint of $\overline{AR}$, $\alpha = m(\angle DOS)$, and $\beta = m(\angle EOS)$. Then $OS = \frac{1}{2}$, $SD = \frac{\tan\alpha}{2}$, $SE = \frac{\tan\beta}{2}$, and $\alpha+\beta = 45°$. $DE = SD+SE \to \frac{3}{7} = \frac{\tan\alpha}{2}+\frac{\tan\beta}{2} \to \tan\alpha+\tan\beta = \frac{6}{7}$. Since $\tan\beta = \tan(45° - \alpha) = \frac{1-\tan\alpha}{1+\tan\alpha}$, we have that $\tan\alpha + \frac{1-\tan\alpha}{1+\tan\alpha} = \frac{6}{7}$. Let $w = \tan\alpha$, so $w+\frac{1-w}{1+w} = \frac{6}{7} \to 7w(1+w)+7(1-w) = 6(1+w) \to 7w^2+7 = 6(1+w) \to 7w^2-6w+1 = 0$. This equation has solutions $w = \frac{3\pm\sqrt{2}}{7}$. For ER to be as large as possible, $SE = \frac{\tan\beta}{2}$ should be as small as possible, so $\tan\beta = \frac{3-\sqrt{2}}{7} \to ER = SR - SE = \frac{1}{2} - \frac{3-\sqrt{2}}{14} = \frac{4+\sqrt{2}}{14}$.

Relay Round Solutions

R1-1 If A, B, C, and D are (not necessarily distinct) digits such that $\underline{A}\ \underline{B}\ \underline{C}\ \underline{D} + \underline{A}\ \underline{B}\ \underline{C} + \underline{A}\ \underline{B} + \underline{A} = 2015$. Compute $(A+C)(B+D)$.

R1-1 Solution The sum in the problem equals $1111A+111B+11C+D$. A must be 1, so $111B+11C+D = 2015-1111 = 904$. B must be eight, so $11C+D = 904-888 = 16$. By similar arguments, $C = 1$ and $D = 5$, so $(A+C)(B+D) = (1+1)(5+8) = 26$.

R1-2 Let $T = TNYWR$. Compute the number of positive integers less than or equal to 2015 that are relatively prime to T.

R1-2 Solution $T = 26 = 2 \times 13$. There are $2015/13 = 155$ multiples of 13 between 1 and 2015, and $\lfloor 2015/2 \rfloor = 1007$ multiples of 2. Of the multiples of 13, 77 of them are multiples of 2 as well, so there are $1007 + 155 - 77 = 1085$ numbers that are not relatively prime to 26, so there are $2015 - 1085 = 930$ integers between 1 and 2015 that are relatively prime to 26.

R2-1 The positive integer N can be written in the form $p^a q^b$, where p and q are distinct primes, and a and b are positive integers. If the number $25N$ is a perfect square with exactly 27 positive divisors, compute the smallest possible value of N.

R2-1 Solution Provided p and q are neither 5, $\sigma(25p^a q^b) = \sigma(25)\sigma(p^a)\sigma(q^b) = 3(a+1)(b+1)$. N will be minimized when $p = a = b = 2$ and $q = 3$, so $N = 6^2 = 36$.

R2-2 Let $T = TNYWR$. A positive integer is *small-prime-free* if it is not a multiple of 2, 3, or 5. Compute the T^{th} smallest small-prime-free integer.

R2-2 Solution A number is small-prime-free iff it is relatively prime to 30. In the range of integers between $30k+1$ and $30k+30$ for $k \geq 0$, there are $\phi(30) = 30 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 8$ numbers relatively prime to 30 between $30k+1$ and $30k+30$, inclusive. Accordingly, the 36^{th} small-prime-free number is the fourth such number between 121 and 150 inclusive. The numbers less than 30 that are relatively prime to 30 are $\{1, 7, 11, 13, 17, 19, 23, 29\}$, so the answer is $120 + 13 = 133$.

R2-3 Let $T = TNYWR$. The product of two consecutive positive odd integers is $3T$. Compute the average of these two integers.

R2-3 Solution If the integers are $2n-1$ and $2n+1$, $(2n-1)(2n+1) = 4n^2 - 1 = 3T = 399 \to 4n^2 = 400 \to n = 10$. The two numbers are 19 and 21 and their average is 20.

R3-1 $\frac{1}{2} - \frac{1}{3} = \frac{1}{7} + \frac{1}{N}$. Compute N.

R3-1 Solution The left side difference is $\frac{1}{6}$ and $\frac{1}{6}-\frac{1}{7}=\frac{1}{42}$, so $N=42$.

R3-2 Let $T = TNYWR$. There are k competitors in a contest. There are T ways to select a winner and a runner-up. Compute k.

R3-2 Solution There are $k(k-1)$ ways to pick a winner and runner up. Since $T=42$, $k=7$.

R3-3 Let $T = TNYWR$. A positive integer is a palindrome if its digits read the same forwards as backwards. There are exactly 20 integers between 1 and N inclusive that are palindromes when written in base T. Compute the smallest (base-10) value of N.

R3-3 Solution Single digits $(\underline{a})_T$ are palindromes, as are numbers of the form $(\underline{a}\ \underline{a})_T$, $(\underline{a}\ \underline{b}\ \underline{a})_T$, and so forth. Because $T=7$, There are 6 singleton and doubleton palindromes and 7 three-digit palindromes that begin with 1, which makes 19 in total. The first 3-digit palindrome that starts with a 2 is $(202)_7 = 100$.

R3-4 Let $T = TNYWR$. Compute the smallest value of n such that the product of any n consecutive integers is a multiple of T.

R3-4 Solution $T = 100$. In order for the product of n consecutive integers to be a multiple of 100, at least two of the integers must be multiples of 5. The smallest interval of consecutive integers that is guaranteed to contain two multiples of 5 is contains 10 integers.

R3-5 Let $T = TNYWR$. There are circles of radius T with centers A, R, M, and L. $ARML$ is square of side length $2T$. If x is the area that is inside $ARML$ but outside any of the circles, compute x to the nearest tenth. Pass back your answer in decimal form.

R3-5 Solution Note that a circle of radius T with center at the center of ARML would have the same amount of area outside of it as what is described in the problem. Accordingly, as $T=10$, the area outside the circles is $400-100\pi \approx 400-100(3.1416)=85.8$.

R3-6 Let $T = TNYWR$. A book that was originally priced K dollars is on sale for T dollars, a discount of D percent. If K and D are both positive integers, compute the minimum value of D.

R3-6 Solution $\frac{T}{1-\frac{D}{100}} = K \to T = K - \frac{KD}{100} \to 100T = 100K - KD \to 100 - \frac{100T}{K} = D$. To get the minimum positive integral value of D, $\frac{100T}{K}$ must be the largest factor of $100T$ less than 100. For $T = 85.8$, $100T = 8580 = 2^2 \times 3 \times 5 \times 11 \times 13$. The largest product of factors of 8580 less than 100 is 78, so $D = 22$.

Tiebreaker Solution

TB Compute the sum of the integral values of x such that $x^2 + 60x + 1000$ is a perfect square.

TB Solution $x^2 + 60x + 1000 = (x + 30)^2 + 100$. If it is equal to a perfect square, say y^2 with $y \geq 0$, then $y^2 - (x+30)^2 = 100 \rightarrow (y - (x+30))(y + (x+30)) = 100$. Note that the two factors will have identical parity, so either they are both equal to 10, in which case $x = -30$ and $y = 10$, or one is equal to 2 and the other is equal to 50. If the first factor is equal to 2 then $x = -6$ and $y = 26$. If the first factor is equal to 50 then $x = -54$ and $y = 26$. The sum of the values of x is $(-30) + (-6) + (-54) = -90$.